

A stochastic approach to path–dependent nonlinear Kolmogorov equations via BSDEs with time–delayed generators and applications to finance

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April 5, 2016

Abstract

We prove the existence of a viscosity solution of the following path dependent nonlinear Kolmogorov equation:

$$\begin{cases} \partial_t u(t, \phi) + \mathcal{L}u(t, \phi) + f(t, \phi, u(t, \phi), \partial_x u(t, \phi) \sigma(t, \phi), (u(\cdot, \phi))_t) = 0, & t \in [0, T], \phi \in \mathbb{A}, \\ u(T, \phi) = h(\phi), & \phi \in \mathbb{A}, \end{cases}$$

where $\mathbb{A} = \mathcal{C}([0, T]; \mathbb{R}^d)$, $(u(\cdot, \phi))_t := (u(t + \theta, \phi))_{\theta \in [-\delta, 0]}$ and

$$\mathcal{L}u(t, \phi) := \langle b(t, \phi), \partial_x u(t, \phi) \rangle + \frac{1}{2} \text{Tr}[\sigma(t, \phi) \sigma^*(t, \phi) \partial_{xx}^2 u(t, \phi)].$$

The result is obtained by a stochastic approach. In particular we prove a new type of nonlinear Feynman–Kac representation formula associated to a backward stochastic differential equation with time–delayed generator which is of non–Markovian type. Applications to the large investor problem and risk measures via g –expectations are also provided.

AMS Classification subjects: 35D40, 35K10, 60H10, 60H30

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Keywords or phrases: Path-dependent partial differential equations; viscosity solutions; Feynman–Kac formula; backward stochastic differential equations; time-delayed generators.

1 Introduction

We aim at providing a probabilistic representation of a viscosity solution to the following path-dependent nonlinear Kolmogorov equation (PDKE)

$$\begin{cases} -\partial_t u(t, \phi) - \mathcal{L}u(t, \phi) - f(t, \phi, u(t, \phi), \partial_x u(t, \phi) \sigma(t, \phi), (u(\cdot, \phi))_t) = 0, \\ u(T, \phi) = h(\phi), \end{cases} \quad (1)$$

for $t \in [0, T]$, $\phi \in \mathbb{A} := \mathcal{C}([0, T]; \mathbb{R}^d)$ being the space of continuous \mathbb{R}^d -valued functions defined on the interval $[0, T]$, being $T < \infty$ a fixed time horizon. Also, for a fixed delay $\delta > 0$, we have set $(u(\cdot, \phi))_t := (u(t + \theta, \phi))_{\theta \in [-\delta, 0]}$. In equation (1) we have denoted by \mathcal{L} the second order differential operator given by

$$\mathcal{L}u(t, \phi) := \frac{1}{2} \text{Tr}[\sigma(t, \phi) \sigma^*(t, \phi) \partial_{xx}^2 u(t, \phi)] + \langle b(t, \phi), \partial_x u(t, \phi) \rangle,$$

with $b : [0, T] \times \mathbb{A} \rightarrow \mathbb{R}^d$ and $\sigma : [0, T] \times \mathbb{A} \rightarrow \mathbb{R}^{d \times d'}$ being two non-anticipative functionals to be better introduced in subsequent section.

In particular we will prove that, under appropriate assumptions on the coefficients, being

$$(X^{t, \phi}(s), Y^{t, \phi}(s), Z^{t, \phi}(s))_{s \in [t, T]},$$

the unique solution to the decoupled forward-backward stochastic differential system

$$\begin{cases} X^{t, \phi}(s) = \phi(t) + \int_t^s b(r, X^{t, \phi}) dr + \int_t^s \sigma(r, X^{t, \phi}) dW(r), & s \in [t, T], \\ Y^{t, \phi}(s) = h(X^{t, \phi}) + \int_s^T f(r, X^{t, \phi}, Y^{t, \phi}(r), Z^{t, \phi}(r), Y_r^{t, \phi}) dr \\ \quad - \int_s^T Z^{t, \phi}(r) dW(r), & s \in [t, T], \end{cases} \quad (2)$$

with $(t, \phi) \in [0, T] \times \mathbb{A}$ and W a standard Brownian motion, then the deterministic non-anticipative functional $u : [0, T] \times \mathbb{A} \rightarrow \mathbb{R}$ given by the representation formula $u(t, \phi) := Y^{t, \phi}(t)$ is a viscosity solution, in the sense of [15], to equation (1). Above, the notation $Y_r^{t, \phi}$ appearing in the generator f of the backward component in system (2) stands for the path of the process $Y^{t, \phi}$ restricted to $[r - \delta, r]$, namely

$$Y_r^{t, \phi} := (Y^{t, \phi}(r + \theta))_{\theta \in [-\delta, 0]}.$$

In particular the forward equation is a functional stochastic differential equation, while the backward equation has time-delayed generator, that is the generator f can depend, unlike the classical backward stochastic differential equations, on the past values of $Y^{t, \phi}$.

Let us stress that if we do not consider delay neither in the forward nor in the backward component, we retrieve standard results of Markovian forward-backward system, so that in this case we obtain $u(t, \phi) = u(t, \phi(t))$, and equation (1) becomes

$$\begin{cases} -\partial_t u(t, x) - \mathcal{L}u(t, x) - f(t, x, u(t, x), \partial_x u(t, x) \sigma(t, x)) = 0, & t \in [0, T), x \in \mathbb{R}^d, \\ u(T, x) = h(x), & x \in \mathbb{R}^d, \end{cases}$$

with

$$\mathcal{L}u(t, x) := \frac{1}{2} \text{Tr}[\sigma(t, x) \sigma^*(t, x) \partial_{xx}^2 u(t, x)] + \langle b(t, x), \partial_x u(t, x) \rangle.$$

Let us recall that BSDE's were first introduced, in the linear case, by Bismut [4], whereas the nonlinear case was considered by Pardoux and Peng in [29]. Later, in [30, 36], the connection between BSDEs and semilinear parabolic partial differential equations (PDE's) was established, proving the nonlinear Feynman-Kac formula for Markovian equations stated above. Also, a similar deterministic representation associated with a suitable PDE, can be proved taking into account different types of BSDE's, such as BSDE's with random terminal time, see, e.g. [8], reflected BSDE's, see, e.g. [18], or also backward stochastic variational inequalities, see, e.g. [25, 26].

When one is to consider the non-Markovian case, the associated PDE becomes path-dependent. In particular in [33] the author shows for the first time that a non-Markovian BSDE can be linked with a path-dependent PDE. Subsequently in [37] the authors proved, in the case of smooth coefficients, the existence and uniqueness of a classical solution for the path-dependent Kolmogorov equation

$$\begin{cases} -\partial_t u(t, \phi) - \frac{1}{2} \partial_{xx}^2 u(t, \phi) - f(t, \phi, u(t, \phi), \partial_x u(t, \phi)) = 0, & t \in [0, T), \phi \in \mathbb{A}, \\ u(T, \phi) = h(\phi), & \phi \in \mathbb{A}. \end{cases} \quad (3)$$

In particular the authors appealed to a representation formula using the standard non-Markovian BSDE:

$$Y^{t, \phi}(s) = h(W^{t, \phi}) + \int_s^T f(W^{t, \phi}, Y^{t, \phi}(r), Z^{t, \phi}(r)) dr - \int_s^T Z^{t, \phi}(r) dW(r), \quad s \in [t, T], \quad (4)$$

with the generator and the final condition depending on the Brownian paths: $W^{t, \phi}(s) = \phi(t) + W(s) - W(t)$, if $s \in [t, T]$ and $W^{t, \phi}(s) = \phi(s)$, if $s \in [0, t]$. Then in [35] a new type of viscosity solution is introduced.

Eventually in [15, 16, 17] the authors introduced a new notion of viscosity solutions, which is the definition we will consider in the present work, for semilinear and fully non linear path-dependent PDE, using the framework of functional Itô calculus first set by Dupire [14] and Cont & Fournié [6].

We will, in the present work, generalized the results in [15] along two directions. First we will consider a BSDE whose generator depends not only on past valued assumed by a standard Brownian motion W , but the BSDE may depends on a general diffusion process X . Second, and most important generalization, we will prove the connection between path-dependent PDEs and BSDEs with time-delayed generators. We recall that time-delayed

BSDE were first introduced in [12] and [13]. More precisely the authors obtained the existence and uniqueness of the solution the the time-delayed BSDE

$$Y(t) = \xi + \int_t^T f(s, Y_s, Z_s) ds - \int_t^T Z(s) dW(s), \quad 0 \leq t \leq T, \quad (5)$$

where $Y_s := (Y(r))_{r \in [0, s]}$ and $Z_s := (Z(r))_{r \in [0, s]}$. In particular, the aforementioned existence and uniqueness result holds true if the time horizon T or the Lipschitz constant for the generator f are sufficiently small. To our best knowledge, the link between time-delayed BSDE's and path-dependent PDE's has never been addressed in literature.

We also emphasize that our framework, since our BSDE is time-delayed, requires that the backward equation contains a supplementary initial condition to be satisfied, namely $Y^{t, \phi}(s) = Y^{s, \phi}(s)$, if $s \in [0, t]$. Let us further stress that the Feynman-Kac formula would fail with standard prolongation $Y^{t, \phi}(s) = Y^{t, \phi}(t)$, for $s \in [0, t]$. Although the existence results for equation (5) has already been treated in [12, 13], this new initial condition imposes a more elaborated proof.

The last part of the paper presents two financial models based on our theoretical results. In recent years delay equations have been of growing interest, mainly motivated by many concrete applications where the effect of delay cannot be neglected, see, e.g. [27, 39]. On the contrary, BSDEs with time delayed generator have been first introduced as a pure mathematical tool, with no application of interest. Only later in [9, 10] the author proposed some financial applications to pricing, hedging and investment portfolio management, where backward equations with delayed generator provide a fundamental tool.

Based on the recently introduced path-dependent calculus, together with the mild assumptions of differentiability required, the probabilistic representation for a viscosity solution of a non-linear parabolic equations proved in the present paper, finds perhaps its best application in finance. In fact, a wide variety of financial derivatives can be formally treated under the theory developed in what follows, from the more standard European options, to the more exotic path-dependent options, such as Asian options or look-back options.

We propose here two possible applications of forward-backward stochastic differential system (2), where the delay in the backward component arises from two different motivations. The first example we will deal with is a generalization of a well-known model in finance, where we will consider the case of a non standard investor acting on a financial market. We will assume, following [7, 20], that a so called *large investor* wishes to invest on a given market, buying or selling a stock. This investor has the peculiarity that his actions on the market can affect the stock price. In particular, we will assume that the stock price S and the bond B are a function of the large investor's portfolio (X, π) , X being the value of the portfolio and π the number of share of the asset S .

This case has been already treated in financial literature, see, e.g. [20]. We further generalize the aforementioned results assuming a second market imperfection, that is we assume that it might be a small time delay between the action of the large investor and the reaction of the market, so that we are led to consider the financial system with the presence of the

past of X in the coefficients r, μ and σ :

$$\begin{cases} \frac{dB(t)}{B(t)} = r(t, X(t), \pi(t), X_t) dt, & B(0) = 1, \\ \frac{dS(t)}{S(t)} = \mu(t, X(t), \pi(t), X_t) dt + \sigma(t, X(t), X_t) dW(t), & S(0) = s_0 > 0, \end{cases}$$

where the notation X_t stands for the path $(X(t + \theta))_{\theta \in [-\delta, 0]}$, being δ a small enough delay.

The second example we deal with arises from a different situation. Recent literature in financial mathematics has been focused in how to measure the riskiness of a given financial investment. To this extent *dynamic risk measures* have been introduced in [2]. In particular, BSDEs have been shown to be perhaps the best mathematical tool for modelling dynamic risk measures, via the so called *g-expectations*. In [10] the author proposed a risk measure that takes also into account the past values assumed by the investment, that is we will assume that, in making his future choices, the investor will consider not only the present value of the investment, but also the values assumed in a sufficiently small past interval. This has been motivated by empirical studies that show how the memory effect has a fundamental importance in an investor's choices, see, e.g. [10] and references therein for financial studies on the memory effect in financial investment. We therefore consider an investor that tries to quantify the riskiness of a given investment, with Y being his investment, we will assume that the investor looks at the average value of his investment in a sufficiently recent past, that is we consider a generator of the form $\frac{1}{\delta} \int_{-\delta}^0 Y(t + \theta) d\theta$, with $\delta > 0$ being a sufficiently small delay.

The paper is organized as follows: in Section 2 we introduce needed notion based on functional Itô's calculus and the notion of viscosity solution for path-dependent PDE's. In Section 3 we prove the existence and uniqueness of a solution for the time-delayed BSDE, whereas Section 4 is devoted to the main results of the present work, that is the proof of the continuity of the function $u(t, \phi) := Y^{t, \phi}(t)$ as well as the generalization of the Feynman-Kac formula with the core of the present work, that is Theorem 11. Eventually in Section 5 we present the financial applications.

2 Preliminaries

2.1 Pathwise derivatives and functional Itô's formula

Let us first introduce the framework on which we shall construct the solutions of PDKE (1). In particular for a deep treatment of functional Itô calculus we refer the reader to Dupire [14] and Cont & Fournié [6].

Let $\hat{\Lambda} := \mathbb{D}([0, T]; \mathbb{R}^d)$ be the set of càdlàg \mathbb{R}^d -valued functions, i.e. right continuous, with finite left-hand limits, \hat{B} the canonical process on $\hat{\Lambda}$, i.e. $\hat{B}(t, \hat{\phi}) := \hat{\phi}(t)$ and $\hat{\mathbb{F}} := (\hat{\mathcal{F}}_s)_{s \in [0, T]}$ the filtration generated by \hat{B} . On $\hat{\Lambda}$, resp. $[0, T] \times \hat{\Lambda}$, we introduce the following norm, resp. pseudometric, with respect to whom it becomes a Banach space, resp. a complete pseudometric space. Let us thus define, for any $(t, \hat{\phi}), (t', \hat{\phi}') \in [0, T] \times \hat{\Lambda}$,

$$\begin{aligned} \|\hat{\phi}\|_T &:= \sup_{r \in [0, T]} |\hat{\phi}(r)|, \\ d((t, \hat{\phi}), (t', \hat{\phi}')) &:= |t - t'| + \sup_{r \in [0, T]} |\hat{\phi}(r \wedge t) - \hat{\phi}'(r \wedge t')|. \end{aligned}$$

Let $\hat{u} : [0, T] \times \hat{\Lambda} \rightarrow \mathbb{R}$ be an $\hat{\mathbb{F}}$ -progressively measurable non-anticipative process, that is we assume $\hat{u}(t, \hat{\phi})$ depends only on the restriction of $\hat{\phi}$ on $[0, t]$, i.e. $\hat{u}(t, \hat{\phi}) = \hat{u}(t, \hat{\phi}(\cdot \wedge t))$, for any $(t, \hat{\phi}) \in [0, T] \times \hat{\Lambda}$. We say that \hat{u} is vertically differentiable at $(t, \hat{\phi}) \in [0, T] \times \hat{\Lambda}$ if there exist

$$\partial_{x_i} \hat{u}(t, \hat{\phi}) := \lim_{h \rightarrow 0} \frac{\hat{u}(t, \hat{\phi} + h \mathbb{1}_{[t, T]} e_i) - \hat{u}(t, \hat{\phi})}{h}$$

for any $i = \overline{1, d}$, where we have denoted by $\{e_i\}_{i=\overline{1, d}}$ the canonical basis of \mathbb{R}^d . The second order derivatives, when they exist, are denoted by $\partial_{x_i x_j}^2 \hat{u}(t, \hat{\phi}) := \partial_{x_i}(\partial_{x_j} \hat{u})$, for any $i, j = \overline{1, d}$. Let us further denote by $\partial_x \hat{u}(t, \hat{\phi})$ the gradient vector, that is we have

$$\partial_x \hat{u}(t, \hat{\phi}) = \left(\partial_{x_1} \hat{u}(t, \hat{\phi}), \dots, \partial_{x_d} \hat{u}(t, \hat{\phi}) \right),$$

and by $\partial_{xx}^2 \hat{u}(t, \hat{\phi})$ the $d \times d$ -Hessian matrix, that is

$$\partial_{xx}^2 \hat{u}(t, \hat{\phi}) = \left(\partial_{x_i x_j}^2 \hat{u}(t, \hat{\phi}) \right)_{i, j = \overline{1, d}}.$$

Let $t \in [0, T]$ and a path $\phi \in \hat{\Lambda}$, we denote

$$\phi_{(t)} := \phi(\cdot \wedge t) \in \hat{\Lambda}. \quad (6)$$

We say that \hat{u} is horizontally differentiable at $(t, \hat{\phi}) \in [0, T] \times \hat{\Lambda}$ if there exist

$$\partial_t \hat{u}(t, \hat{\phi}) := \lim_{h \rightarrow 0+} \frac{\hat{u}(t+h, \hat{\phi}_{(t)}) - \hat{u}(t, \hat{\phi})}{h},$$

for $t \in [0, T]$ and $\partial_t \hat{u}(T, \hat{\phi}) := \lim_{t \rightarrow T-} \partial_t \hat{u}(t, \hat{\phi})$.

Let $\hat{u} : [0, T] \times \hat{\Lambda} \rightarrow \mathbb{R}$ be non-anticipative, we say that $\hat{u} \in \mathcal{C}([0, T] \times \hat{\Lambda})$ if \hat{u} is continuous on $[0, T] \times \hat{\Lambda}$ under the pseudometric d ; we write that $\hat{u} \in \mathcal{C}_b([0, T] \times \hat{\Lambda})$ if $\hat{u} \in \mathcal{C}([0, T] \times \hat{\Lambda})$ and \hat{u} is bounded on $[0, T] \times \hat{\Lambda}$. Eventually we say that $\hat{u} \in \mathcal{C}_b^{1,2}([0, T] \times \hat{\Lambda})$ if $\hat{u} \in \mathcal{C}([0, T] \times \hat{\Lambda})$ and the derivatives $\partial_x \hat{u}$, $\partial_{xx}^2 \hat{u}$, $\partial_t \hat{u}$ exist and they are continuous and bounded.

Having introduced the needed notations, following [14], we will now work with processes u defined on $[0, T] \times \Lambda \rightarrow \mathbb{R}$, being Λ the space of continuous paths, $\mathcal{C}([0, T]; \mathbb{R}^d)$. Let B be the canonical process on Λ , i.e. $B(t, \phi) := \phi(t)$ and $\mathbb{F} := (\mathcal{F}_s)_{s \in [0, T]}$ the filtration generated by B .

From the fact that Λ is a closed subspace of $\hat{\Lambda}$, we have that $(\Lambda, \|\cdot\|_t)$ is also a Banach space; with an analogous reasoning we claim that $([0, T] \times \Lambda, d)$ is a complete pseudometric space. As done above, we have that if $u : [0, T] \times \Lambda \rightarrow \mathbb{R}$ is a non-anticipative process, we write that $u \in \mathcal{C}([0, T] \times \Lambda)$ if u is continuous on $[0, T] \times \Lambda$ under the pseudometric d ; if, moreover, u is continuous and bounded on $[0, T] \times \Lambda$, we write $u \in \mathcal{C}_b([0, T] \times \Lambda)$. Eventually, following [15], we write that $u \in \mathcal{C}_b^{1,2}([0, T] \times \Lambda)$ if there exists $\hat{u} \in \mathcal{C}_b^{1,2}([0, T] \times \hat{\Lambda})$ such that $\hat{u}|_{[0, T] \times \Lambda} = u$ and by definition we take $\partial_t u := \partial_t \hat{u}$, $\partial_x u := \partial_x \hat{u}$, $\partial_{xx}^2 u := \partial_{xx}^2 \hat{u}$, notice that definitions are independent of the choice of \hat{u} .

We are now to introduce the shifted spaces of càdlàg and continuous paths. If $t \in [0, T]$, \hat{B}^t is the shifted canonical process on $\hat{\Lambda}^t := \mathbb{D}([t, T]; \mathbb{R}^d)$, $\hat{\mathbb{F}}^t := (\hat{\mathcal{F}}_s^t)_{s \in [t, T]}$ is the shifted

filtration generated by \hat{B}^t ,

$$\begin{aligned} \|\hat{\phi}\|_T^t &:= \sup_{r \in [t, T]} |\hat{\phi}(r)|, \\ d^t((s, \hat{\phi}), (s', \hat{\phi}')) &:= |s - s'| + \sup_{r \in [t, T]} |\hat{\phi}(r \wedge s) - \hat{\phi}'(r \wedge s')|, \end{aligned}$$

for any $(s, \hat{\phi}), (s', \hat{\phi}') \in [t, T] \times \hat{\mathbb{A}}^t$. Analogously we define the spaces $\mathcal{C}([t, T] \times \hat{\mathbb{A}}^t)$, $\mathcal{C}_b([t, T] \times \hat{\mathbb{A}}^t)$ and $\mathcal{C}_b^{1,2}([t, T] \times \hat{\mathbb{A}}^t)$. Similarly, we denote $\mathbb{A}^t := \mathcal{C}([t, T]; \mathbb{R}^d)$, B^t the shifted canonical process on \mathbb{A}^t , $\mathbb{F}^t := (\mathcal{F}_s^t)_{s \in [t, T]}$ the shifted filtration generated by B^t and we introduce the spaces $\mathcal{C}([t, T] \times \mathbb{A}^t)$, $\mathcal{C}_b([t, T] \times \mathbb{A}^t)$ and $\mathcal{C}_b^{1,2}([t, T] \times \mathbb{A}^t)$.

Let us denote by \mathcal{T} the set of all \mathbb{F} -stopping times τ such that for all $t \in [0, T]$, then we have that the set $\{\phi \in \mathbb{A} : \tau(\phi) > t\}$ is an open subset of $(\mathbb{A}, \|\cdot\|_T)$ and \mathcal{T}^t the be the set of all \mathbb{F} -stopping times τ such that for all $s \in [t, T]$, the set $\{\phi \in \mathbb{A}^t : \tau(\phi) > s\}$ is an open subset of $(\mathbb{A}^t, \|\cdot\|_T^t)$.

For a càdlàg function $\phi \in \mathbb{D}([-\delta, T]; \mathbb{R}^d)$, we denote

$$\phi_t := (\phi(t + \theta))_{\theta \in [-\delta, 0]}. \quad (7)$$

We conclude this subsection by recalling the functional version of the Itô's formula (see Cont & Fournié [6, Theorem 4.1]).

Theorem 1 (Functional Itô's formula) *Let A be a d -dimensional Itô process, i.e. $A : [0, T] \times \mathbb{A} \rightarrow \mathbb{R}^d$ is a continuous \mathbb{R}^d -valued semimartingale defined on the probability space $(\mathbb{A}, \mathbb{F}, \mathbb{P})$ which admits the representation*

$$A(t) = A(0) + \int_0^t b(r) dr + \int_0^t \sigma(r) dB(r), \quad \text{for all } t \in [0, T].$$

If $F \in \mathcal{C}_b^{1,2}([0, T] \times \hat{\mathbb{A}})$ then, for any $t \in [0, T]$, the following change of variable formula holds true:

$$\begin{aligned} F(t, A_{(t)}) &= F(0, A_{(0)}) + \int_0^t \partial_t F(r, A_{(r)}) dr + \int_0^t \langle \partial_x F(r, A_{(r)}), b(r) \rangle dr \\ &\quad + \frac{1}{2} \int_0^t \text{Tr}[\sigma(r) \sigma^*(r) \partial_{xx}^2 F(r, A_{(r)})] dr + \int_0^t \langle \partial_x F(r, A_{(r)}), \sigma(r) dB(r) \rangle. \end{aligned}$$

2.2 Path-dependent PDEs

We are now to introduced the notion of viscosity solution to equation (1), in particular we will use the notion of viscosity solution first introduced in [15], see also [16, 17].

Let $(t, \phi) \in [0, T] \times \mathbb{A}$ be fixed and $(W(t))_{t \geq 0}$ be a d' -dimensional standard Brownian motion defined on some complete probability space $(\Omega, \mathcal{G}, \mathbb{P})$. We denote by $\mathbb{G}^t = (\mathcal{G}_s^t)_{s \in [0, T]}$ the natural filtration generated by $((W(s) - W(t))\mathbb{1}_{\{s \geq t\}})_{s \in [0, T]}$ and augmented by the set of \mathbb{P} -null events of \mathcal{G} .

Let us take $L \geq 0$ and $t < T$. We denote by \mathcal{U}_t^L the space of \mathbb{G}^t -progressively measurable \mathbb{R}^d -valued processes λ such that $|\lambda| \leq L$. We define a new probability measure $\mathbb{P}^{t, \lambda}$ by $d\mathbb{P}^{t, \lambda} := M^{t, \lambda}(T) d\mathbb{P}$, where

$$M^{t, \lambda}(s) := \exp \left(\int_t^s \lambda(r) dW(r) - \frac{1}{2} \int_t^s |\lambda(r)|^2 dr \right), \quad \mathbb{P}\text{-a.s.}$$

Under some suitable assumptions on the coefficients, to be better specified later on, see also Theorem 4 in what follows, the existence and uniqueness of a continuous and adapted stochastic process $(X^{t,\phi}(s))_{s \in [0,T]}$ such that

$$\begin{cases} X^{t,\phi}(s) = \phi(t) + \int_t^s b(r, X^{t,\phi})dr + \int_t^s \sigma(r, X^{t,\phi})dW(r), & s \in [t, T], \\ X^{t,\phi}(s) = \phi(s), & s \in [0, t), \end{cases}$$

where $(t, \phi) \in [0, T] \times \mathbb{A}$ is given.

We are now ready to define the space of the test functions,

$$\begin{aligned} \underline{A}^L u(t, \phi) &:= \left\{ \varphi \in \mathcal{C}_b^{1,2}([0, T] \times \mathbb{A}) : \exists \tau_0 \in \mathcal{T}_+^t, \varphi(t, \phi) - u(t, \phi) \right. \\ &\quad \left. = \min_{\tau \in \mathcal{T}^t} \underline{\mathcal{E}}_t^L[(\varphi - u)(\tau \wedge \tau_0, X^{t,\phi})] \right\} \end{aligned}$$

and

$$\begin{aligned} \overline{A}^L u(t, \phi) &:= \left\{ \varphi \in \mathcal{C}_b^{1,2}([0, T] \times \mathbb{A}) : \exists \tau_0 \in \mathcal{T}_+^t, \varphi(t, \phi) - u(t, \phi) \right. \\ &\quad \left. = \max_{\tau \in \mathcal{T}^t} \overline{\mathcal{E}}_t^L[(\varphi - u)(\tau \wedge \tau_0, X^{t,\phi})] \right\}, \end{aligned}$$

where $\mathcal{T}_+^t := \{\tau \in \mathcal{T}^t : \tau > t\}$, if $t < T$ and $\mathcal{T}_+^T := \{T\}$. Also, for any $\xi \in L^2(\mathcal{F}_T^t; \mathbb{P})$, $\underline{\mathcal{E}}_t^L(\xi) := \inf_{\lambda \in \mathcal{U}_t^L} xxx \mathbb{E}^{\mathbb{P}^{t,\lambda}}(\xi)$ and $\overline{\mathcal{E}}_t^L(\xi) := \sup_{\lambda \in \mathcal{U}_t^L} \mathbb{E}^{\mathbb{P}^{t,\lambda}}(\xi)$ are nonlinear expectations.

We are now able to give the definition of a viscosity solution of the functional PDE (1), see, e.g. [15, Def. 3.3].

Definition 2 Let $u \in \mathcal{C}_b([0, T] \times \mathbb{A})$ such that $u(T, \phi) = h(\phi)$, for all $\phi \in \mathbb{A}$.

(a) For any $L \geq 0$, we say that u is a viscosity L -subsolution of (1) if at any point $(t, \phi) \in [0, T] \times \mathbb{A}$, for any $\varphi \in \underline{A}^L u(t, \phi)$, it holds

$$-\partial_t \varphi(t, \phi) - \mathcal{L}\varphi(t, \phi) - f(t, \phi, \varphi(t, \phi), \partial_x \varphi(t, \phi) \sigma(t, \phi), (\varphi(\cdot, \phi))_t) \leq 0.$$

(b) For any $L \geq 0$, we say that u is a viscosity L -supersolution of (1) if at any point $(t, \phi) \in [0, T] \times \mathbb{A}$, for any $\varphi \in \overline{A}^L u(t, \phi)$, we have

$$-\partial_t \varphi(t, \phi) - \mathcal{L}\varphi(t, \phi) - f(t, \phi, \varphi(t, \phi), \partial_x \varphi(t, \phi) \sigma(t, \phi), (\varphi(\cdot, \phi))_t) \geq 0.$$

(c) We say that u is a viscosity subsolution (respectively, supersolution) of (1) if u is a viscosity L -subsolution (respectively, L -supersolution) of (1) for some $L \geq 0$.

(d) We say that u is a viscosity solution of (1) if u is a viscosity subsolution and supersolution of (1).

Remark 3 It is easy to obtain that this definition is equivalent to the classical one in the Markovian framework, see, e.g. [15].

Let us stress that if u is a function from $\mathcal{C}_b^{1,2}([0, T] \times \mathbb{A})$, then it is easy to see that u is a viscosity solution of (1) if and only if u is a classical solution for (1). Indeed, if u is a viscosity solution then $u \in \underline{A}^L u(t, \phi) \cap \overline{A}^L u(t, \phi)$ and therefore u satisfies (1). For the reverse

statement one can use the nonlinear Feynman–Kac formula proved in this new framework, see Theorem 14 below, together with functional Itô’s formula in order to compute $u(s, X^{t,\phi})$ and the existence and uniqueness result for the corresponding stochastic system (2).

Let us also mention that, in accord with standard theory of viscosity solutions, the viscosity property introduced above is a local property, i.e. to check that u is a viscosity solution in (t, ϕ) it is sufficient to know the value of u on the interval $[t, \tau_\epsilon]$, where $\epsilon > 0$ is arbitrarily fixed and $\tau_\epsilon \in \mathcal{T}_+^t$ is given by $\tau_\epsilon := \inf \{s > t : |\phi(s)| \geq \epsilon\} \wedge (t + \epsilon)$.

Eventually, let us notice that since b and σ are Lipschitz, we have uniqueness in law for $X^{t,\phi}$; also, since the filtration on $(\Omega, \mathcal{G}, \mathbb{P})$ is generated by W , every progressively measurable processes λ is a functional of W . Therefore, the spaces of test functions and the above definition are independent on the choice of $(\Omega, \mathcal{G}, \mathbb{P})$ and W .

3 The forward–backward delayed system

We are now to state the existence and uniqueness results for a delayed forward-backward system, where both the forward and the backward component exhibit a delayed behaviour, that is we will assume that the generator of the backward equation may depend on past values assumed by its solution (Y, Z) . In complete generality, since we will need these results in next sections, we will allow the solution to depend on a general initial time and initial values. Also we remark that in order to ensure the existence and uniqueness result, we need to equip the backward with a suitable condition in the time interval $[0, t]$, being t the initial time, fact that implies a different proof than the one provided in [12].

The main goal is to find a family

$$\left(X^{t,\phi}, Y^{t,\phi}, Z^{t,\phi} \right)_{(t,\phi) \in [0,T] \times \mathbb{R}},$$

of stochastic processes such that the following decoupled forward–backward system holds

$$\begin{cases} X^{t,\phi}(s) = \phi(t) + \int_t^s b(r, X^{t,\phi}) dr + \int_t^s \sigma(r, X^{t,\phi}) dW(r), & s \in [t, T], \\ X^{t,\phi}(s) = \phi(s), & s \in [0, t], \\ Y^{t,\phi}(s) = h(X^{t,\phi}) + \int_s^T F(r, X^{t,\phi}, Y^{t,\phi}(r), Z^{t,\phi}(r), Y_r^{t,\phi}, Z_r^{t,\phi}) dr \\ \quad - \int_s^T Z^{t,\phi}(r) dW(r), & s \in [t, T], \\ Y^{t,\phi}(s) = Y^{s,\phi}(s), \quad Z^{t,\phi}(s) = 0, & s \in [0, t]. \end{cases} \quad (8)$$

Let us stress once more, that in both the forward and backward equation, the values of $X^{t,\phi}$, resp. $(Y^{t,\phi}, Z^{t,\phi})$, in the time interval $[0, t]$, resp. $t - \delta, t]$, need to be known; this is one reason we have to impose such initial conditions. The above initial condition for Y is absolutely necessary in view of the Feynman–Kac formula, which will be proven later. We also prolong by convention, $Y^{t,\phi}$ by $Y^{t,\phi}(0)$ on the negative real axis (this is needed in the case that $t < \delta$). For the sake of simplicity, we will take $Z^{t,\phi}(s) := 0$ and $F(s, \cdot, \cdot, \cdot, \cdot, \cdot) := 0$ whenever s become negative.

3.1 The forward path-dependent SDE

Let us first focus on the forward component X appearing in the FBDSDE system (8), next theorem states the existence and the uniqueness, as well as estimates, for the process $(X^{t,\phi}(r))_{r \in [0,T]}$.

The existence result is a classical one (see, e.g. [27] or [28]) and the estimates can be obtained by applying Itô's formula together with assumptions (A_1) – (A_2) , see, e.g. [40], and for these reasons we will not state the proof.

In what follows we will assume the following to hold.

Let us consider two non-anticipative functionals $b : [0, T] \times \mathbb{A} \rightarrow \mathbb{R}^d$ and $\sigma : [0, T] \times \mathbb{A} \rightarrow \mathbb{R}^{d \times d'}$ such that

(A_1) b and σ are continuous;

(A_2) there exists $\ell > 0$ such that for any $t \in [0, T]$, $\phi, \phi' \in \mathbb{A}$,

$$|b(t, \phi) - b(t, \phi')| + |\sigma(t, \phi) - \sigma(t, \phi')| \leq \ell \|\phi - \phi'\|_T.$$

Theorem 4 *Let b, σ satisfying assumptions (A_1) – (A_2) . Let $(t, \phi), (t', \phi') \in [0, T] \times \mathbb{A}$ be given. Then there exists a unique continuous and adapted stochastic process $(X^{t,\phi}(s))_{s \in [0,T]}$ such that*

$$\begin{cases} X^{t,\phi}(s) = \phi(t) + \int_t^s b(r, X^{t,\phi}) dr + \int_t^s \sigma(r, X^{t,\phi}) dW(r), & s \in [t, T], \\ X^{t,\phi}(s) = \phi(s), & s \in [0, t]. \end{cases} \quad (9)$$

Moreover, for any $q \geq 1$, there exists $C = C(q, T, \ell) > 0$ such that

$$\begin{aligned} \mathbb{E}(\|X^{t,\phi}\|_T^{2q}) &\leq C(1 + \|\phi\|_T^{2q}), \\ \mathbb{E}(\|X^{t,\phi} - X^{t',\phi'}\|_T^{2q}) &\leq C\left(\|\phi - \phi'\|_T^{2q} + (1 + \|\phi\|_T^{2q} + \|\phi'\|_T^{2q}) \cdot |t - t'|^q \right. \\ &\quad \left. + \sup_{r \in [t \wedge t', t \vee t']} |\phi(t) - \phi(r)|^{2q}\right), \\ \mathbb{E}\left(\sup_{\substack{s, r \in [t, T] \\ |s-r| \leq \epsilon}} |X^{t,\phi}(s) - X^{t,\phi}(r)|^{2q}\right) &\leq C(1 + \|\phi\|_T^{2q})\epsilon^{q-1}, \quad \text{for all } \epsilon > 0. \end{aligned} \quad (10)$$

3.2 The backward delayed SDE

Let us now consider delayed backward SDE appearing in (8), in particular in what follows we have d and d' are the fixed constants defined above, whereas $m \in \mathbb{N}^*$ is a new fixed constant. Let us then introduce the main reference spaces we will consider.

Definition 5 (i) *let $\mathcal{H}_t^{2, m \times d'}$ denote the space of \mathbb{G}^t -progressively measurable processes $Z : \Omega \times [t, T] \rightarrow \mathbb{R}^{m \times d'}$ such that*

$$\mathbb{E} \left[\int_t^T |Z(s)|^2 ds \right] < \infty;$$

(ii) let $\mathcal{S}_t^{2,m}$ the space of continuous \mathbb{G}^t -progressively measurable processes $Y : \Omega \times [t, T] \rightarrow \mathbb{R}^m$ such that

$$\mathbb{E} \left[\sup_{t \leq s \leq T} |Y(s)|^2 \right] < \infty.$$

Also we will equip the spaces $\mathcal{H}_t^{2,m \times d'}$ and $\mathcal{S}_t^{2,m}$ with the following norms

$$\begin{aligned} \|Z\|_{\mathcal{H}_t^{2,m \times d'}}^2 &= \mathbb{E} \left[\int_t^T e^{\beta s} |Z(s)|^2 ds \right], \\ \|Y\|_{\mathcal{S}_t^{2,m}}^2 &= \mathbb{E} \left[\sup_{t \leq s \leq T} e^{\beta s} |Y(s)|^2 \right], \end{aligned}$$

for a given constant $\beta > 0$.

In what follows, concerning the delayed backward SDE appearing in (8), we will assume the following to hold.

Let

$$F : [0, T] \times \mathbb{A} \times \mathbb{R}^m \times \mathbb{R}^{m \times d'} \times L^2([-\delta, 0]; \mathbb{R}^m) \times L^2([-\delta, 0]; \mathbb{R}^{m \times d'}) \rightarrow \mathbb{R}^m,$$

and

$$h : \mathbb{A} \rightarrow \mathbb{R}^m,$$

such that the following holds:

(A₃) There exist $L, K, M > 0, p \geq 1$ and a probability measure α on $([-\delta, 0], \mathcal{B}([-\delta, 0]))$ such that, for any $t \in [0, T]$, $\phi \in \mathbb{A}$, $(y, z), (y', z') \in \mathbb{R}^m \times \mathbb{R}^{m \times d'}$, $\hat{y}, \hat{y}' \in L^2([-\delta, 0]; \mathbb{R}^m)$ and $\hat{z}, \hat{z}' \in L^2([-\delta, 0]; \mathbb{R}^{m \times d'})$, we have

$$\begin{aligned} (i) \quad & \phi \mapsto F(t, \phi, y, z, \hat{y}, \hat{z}) \text{ is continuous,} \\ (ii) \quad & |F(t, \phi, y, z, \hat{y}, \hat{z}) - F(t, \phi, y', z', \hat{y}, \hat{z})| \leq L(|y - y'| + |z - z'|), \\ (iii) \quad & |F(t, \phi, y, z, \hat{y}, \hat{z}) - F(t, \phi, y, z, \hat{y}', \hat{z}')|^2 \\ & \leq K \int_{-\delta}^0 \left(|\hat{y}(\theta) - \hat{y}'(\theta)|^2 + |\hat{z}(\theta) - \hat{z}'(\theta)|^2 \right) \alpha(d\theta), \\ (iv) \quad & |F(t, \phi, 0, 0, 0, 0)| < M(1 + \|\phi\|_T^p). \end{aligned} \tag{11}$$

(A₄) The function $F(\cdot, \cdot, y, z, \hat{y}, \hat{z})$ is \mathbb{F} -progressively measurable, for any $(y, z, \hat{y}, \hat{z}) \in \mathbb{R}^m \times \mathbb{R}^{m \times d'} \times L^2([-\delta, 0]; \mathbb{R}^m) \times L^2([-\delta, 0]; \mathbb{R}^{m \times d'})$.

(A₅) The function h is continuous and, for all $\phi \in \mathbb{A}$,

$$|h(\phi)| \leq M(1 + \|\phi\|_T^p).$$

Remark 6 In order to show the existence and uniqueness of a solution to the backward part of system (8), we will use a standard Banach's fixed point argument. For that we are obliged to impose that K

or δ should be small enough, see, e.g. restriction (30). Also, in order to obtain the continuity of $Y^{t,\phi}$ with respect to ϕ we are obliged to impose restriction (37).

Hence we will assume the following condition to holds, such that both restrictions hold true, in particular we will assume there exists a constant $\gamma \in (0, 1)$ such that

$$K \frac{\gamma e^{\left(\gamma + \frac{6L^2}{\gamma}\right)\delta}}{(1-\gamma)L^2} \max\{1, T\} < \frac{1}{290}. \quad (12)$$

For K fixed, it seems at first sight that the expression above cannot be made true by letting δ to 0; however, condition (12) will still be verified if we allow L to grow, so we can regard L as a parameter, too.

We are now ready to state the main result of the present section.

Theorem 7 *Let us assume that assumptions (A_3) – (A_5) hold true. If K or δ are small enough, that is they satisfy condition (12), then there exists a unique solution $(Y^{t,\phi}, Z^{t,\phi})_{(t,\phi) \in [0,T] \times \mathbb{A}}$ for the backward stochastic differential system from (8), such that $(Y^{t,\phi}, Z^{t,\phi}) \in \mathcal{S}_t^{2,m} \times \mathcal{H}_t^{2,m \times d'}$, for all $t \in [0, T]$ and $t \mapsto (Y^{t,\phi}, Z^{t,\phi})$ is continuous from $[0, T]$ into $\mathcal{S}_0^{2,m} \times \mathcal{H}_0^{2,m \times d'}$.*

Remark 8 *The main difference between the proof of our result and that of Theorem 2.1 from [12] is due to the supplementary structure condition $Y^{t,\phi}(s) = Y^{s,\phi}(s)$, for $s \in [0, t)$ which should be satisfied by the unknown process $Y^{t,\phi}$.*

We also allow T to be arbitrary and we consider that the time horizon is different from the delay $\delta \in [0, T]$; moreover, we separate the Lipschitz constant L with respect to (y, z) by the Lipschitz constant K with respect to \hat{y} , and hence the restriction (12) can avoid the constant L .

Proof. The existence and uniqueness will be obtained by the Banach fixed point theorem.

Let $\phi \in \mathbb{A}$ be arbitrarily fixed and let us consider the map Γ defined on $\mathcal{A} \times \mathcal{B}$, with $\mathcal{A} := \mathcal{C}([0, T]; \mathcal{S}_0^{2,m})$ and $\mathcal{B} := \mathcal{C}([0, T]; L^2(\Omega; \mathcal{H}_0^{2,m \times d'}))$, in the following way: for $(U, V) \in \mathcal{A} \times \mathcal{B}$, $\Gamma(U, V) = (Y, Z)$, where for $t \in [0, T]$, the couple of adapted processes (Y^t, Z^t) is solution to the equation

$$\begin{cases} Y^t(s) = h(X^{t,\phi}) + \int_s^T F(r, X^{t,\phi}, Y^t(r), Z^t(r), U_r^t, V_r^t) dr - \int_s^T Z^t(r) dW(r), & s \in [t, T], \\ Y^t(s) := Y^s(s), \quad Z^t(s) := 0, & s \in [0, t). \end{cases} \quad (13)$$

Step I.

Let us first show that Γ takes values in the Banach space $\mathcal{A} \times \mathcal{B}$. For that, let us take $(U, V) \in \mathcal{A} \times \mathcal{B}$; we will prove that $(Y, Z) := \Gamma(U, V) \in \mathcal{A} \times \mathcal{B}$, i.e. for every $t \in [0, T]$ we have

$$Y^t \in \mathcal{S}_t^{2,m} \subseteq \mathcal{S}_0^{2,m}, \quad Z^t \in \mathcal{H}_t^{2,m \times d'} \subseteq \mathcal{H}_0^{2,m \times d'} \quad (14)$$

and the applications

$$\begin{aligned} [0, T] \ni t &\mapsto Y^t \in \mathcal{S}_0^{2,m}, \\ [0, T] \ni t &\mapsto Z^t \in \mathcal{H}_0^{2,m \times d'} \end{aligned} \quad (15)$$

are continuous.

Let $t \in [0, T]$ and $t' \in [0, T]$, also, with no loss of generality, we will suppose that $t < t'$ and $t' - t < \delta$.

We have, using (13) that

$$\begin{aligned} & \mathbb{E} \left(\sup_{s \in [0, T]} |Y^t(s) - Y^{t'}(s)|^2 \right) \\ & \leq \mathbb{E} \left(\sup_{s \in [0, t']} |Y^t(s) - Y^{t'}(s)|^2 \right) + \mathbb{E} \left(\sup_{s \in [t', T]} |Y^t(s) - Y^{t'}(s)|^2 \right) \\ & \leq 2\mathbb{E} \left(\sup_{s \in [t, t']} |Y^t(s) - Y^t(t)|^2 \right) + 2\mathbb{E} \left(\sup_{s \in [t, t']} |Y^t(t) - Y^s(s)|^2 \right) \\ & \quad + \mathbb{E} \left(\sup_{s \in [t', T]} |Y^t(s) - Y^{t'}(s)|^2 \right). \end{aligned}$$

From the continuity of the solution of equation (13) with respect to time, we have that

$$\mathbb{E} \left(\sup_{s \in [t, t']} |Y^t(s) - Y^t(t)|^2 \right) \rightarrow 0,$$

as $t' \rightarrow t$.

Concerning the term $\mathbb{E} \left(\sup_{s \in [t', T]} |Y^t(s) - Y^{t'}(s)|^2 \right)$ let us denote for short, only throughout this step,

$$\begin{aligned} \Delta Y(r) &:= Y^t(r) - Y^{t'}(r), \quad \Delta Z(r) := Z^t(r) - Z^{t'}(r) \\ \Delta U(r) &:= U^t(r) - U^{t'}(r), \quad \Delta V(r) := V^t(r) - V^{t'}(r) \end{aligned}$$

and

$$\begin{aligned} \Delta h &:= h(X^{t, \phi}) - h(X^{t', \phi}), \\ \Delta F(r) &:= F(r, X^{t, \phi}, Y^t(r), Z^t(r), U_r^t, V_r^t) - F(r, X^{t', \phi}, Y^{t'}(r), Z^{t'}(r), U_r^{t'}, V_r^{t'}). \end{aligned}$$

Exploiting Itô's formula we have, for any $\beta > 0$ and any $s \in [t, T]$,

$$\begin{aligned} & e^{\beta s} |\Delta Y(s)|^2 + \beta \int_s^T e^{\beta r} |\Delta Y(r)|^2 dr + \int_s^T e^{\beta r} |\Delta Z(r)|^2 dr \\ & = e^{\beta T} |\Delta Y(T)|^2 - 2 \int_s^T e^{\beta r} \langle \Delta Y(r), \Delta Z(r) \rangle dW(r) \\ & \quad + 2 \int_s^T e^{\beta r} \langle \Delta Y(r), F(r, X^{t, \phi}, Y^t(r), Z^t(r), U_r^t, V_r^t) \\ & \quad \quad - F(r, X^{t, \phi}, Y^{t'}(r), Z^{t'}(r), U_r^{t'}, V_r^{t'}) \rangle dr. \end{aligned}$$

From assumptions (A₃)–(A₅), and noting that it holds

$$\begin{aligned}
& \int_s^T e^{\beta r} \left(\int_{-\delta}^0 |\Delta U(r+\theta)|^2 + |\Delta V(r+\theta)|^2 \alpha(d\theta) \right) dr \\
&= \int_{-\delta}^0 \int_s^T e^{\beta r} (|\Delta U(r+\theta)|^2 + |\Delta V(r+\theta)|^2) \alpha(d\theta) dr \\
&= \int_{-\delta}^0 \left(\int_{s+\theta}^{T+\theta} e^{\beta(r-\theta)} |\Delta U(r)|^2 + |\Delta V(r)|^2 dr \right) \alpha(d\theta) \\
&\leq e^{\beta\delta} \cdot \int_{-\delta}^0 \alpha(d\theta) \cdot \int_0^T e^{\beta r} (|\Delta U(r)|^2 + |\Delta V(r)|^2) dr \\
&\leq T e^{\beta\delta} \sup_{r \in [0, T]} (e^{\beta r} |\Delta U(r)|^2) + e^{\beta\delta} \int_0^T e^{\beta r} |\Delta V(r)|^2 dr,
\end{aligned}$$

we have for any $a > 0$,

$$\begin{aligned}
& 2 \int_s^T e^{\beta r} \langle \Delta Y(r), F(r, X^{t, \phi}, Y^t(r), Z^t(r), U_r^t, V_r^t) \\
& \quad - F(r, X^{t, \phi'}, Y^{t'}(r), Z^{t'}(r), U_r^{t'}, V_r^{t'}) \rangle dr \\
&\leq a \int_s^T e^{\beta r} |\Delta Y(r)|^2 + \frac{3}{a} \int_s^T e^{\beta r} |\Delta F(r)|^2 dr + \frac{6L^2}{a} \int_s^T e^{\beta r} (|\Delta Y(r)|^2 + |\Delta Z(r)|^2) dr \\
& \quad + \frac{3TK e^{\beta\delta}}{a} \sup_{r \in [0, T]} (e^{\beta r} |\Delta U(r)|^2) + \frac{3K e^{\beta\delta}}{a} \int_0^T e^{\beta r} |\Delta V(r)|^2 dr.
\end{aligned}$$

Therefore we have

$$\begin{aligned}
& e^{\beta s} |\Delta Y(s)|^2 + \left(\beta - a - \frac{6L^2}{a} \right) \int_s^T e^{\beta r} |\Delta Y(r)|^2 dr + \left(1 - \frac{6L^2}{a} \right) \int_s^T e^{\beta r} |\Delta Z(r)|^2 dr \\
&\leq e^{\beta T} |\Delta Y(T)|^2 + \frac{3}{a} \int_s^T e^{\beta r} |\Delta F(r)|^2 dr - 2 \int_s^T e^{\beta r} \langle \Delta Y(r), \Delta Z(r) \rangle dW(r) \\
& \quad + \frac{3TK e^{\beta\delta}}{a} \sup_{r \in [0, T]} e^{\beta r} |\Delta U(r)|^2 + \frac{3K e^{\beta\delta}}{a} \int_0^T e^{\beta r} |\Delta V(r)|^2 dr.
\end{aligned}$$

We now choose $\beta, a > 0$ such that

$$a + \frac{6L^2}{a} < \beta \quad \text{and} \quad \frac{6L^2}{a} < 1. \quad (16)$$

so that we obtain

$$\begin{aligned}
& \left(1 - \frac{6L^2}{a} \right) \mathbb{E} \int_s^T e^{\beta r} |\Delta Z(r)|^2 dr \leq \mathbb{E}(e^{\beta T} |\Delta h|^2) + \frac{3}{a} \mathbb{E} \int_s^T e^{\beta r} |\Delta F(r)|^2 dr \\
& \quad + \frac{3TK e^{\beta\delta}}{a} \mathbb{E} \left(\sup_{r \in [0, T]} e^{\beta r} |\Delta U(r)|^2 \right) + \frac{3K e^{\beta\delta}}{a} \mathbb{E} \int_0^T e^{\beta r} |\Delta V(r)|^2 dr.
\end{aligned} \quad (17)$$

and, exploiting Burkholder–Davis–Gundy’s inequality, we have that

$$\begin{aligned} & 2\mathbb{E}\left(\sup_{s \in [t', T]} \left| \int_s^T e^{\beta r} \langle \Delta Y(r), \Delta Z(r) \rangle dW(r) \right| \right) \\ & \leq \frac{1}{4}\mathbb{E}\left(\sup_{s \in [t', T]} e^{\beta s} |\Delta Y(s)|^2\right) + 144\mathbb{E}\int_{t'}^T e^{\beta r} |\Delta Z(r)|^2 dr. \end{aligned}$$

which immediately implies

$$\begin{aligned} & \frac{3}{4}\mathbb{E}\left(\sup_{s \in [t', T]} e^{\beta s} |\Delta Y(s)|^2\right) \leq \mathbb{E}(e^{\beta T} |\Delta h|^2) + \frac{3}{a}\mathbb{E}\int_{t'}^T e^{\beta r} |\Delta F(r)|^2 dr \\ & + \frac{3TK e^{\beta \delta}}{a}\mathbb{E}\left(\sup_{r \in [0, T]} e^{\beta r} |\Delta U(r)|^2\right) + \frac{3K e^{\beta \delta}}{a}\mathbb{E}\int_0^T e^{\beta r} |\Delta V(r)|^2 dr \\ & + 144\mathbb{E}\int_{t'}^T |\Delta Z(r)|^2 dr. \end{aligned}$$

Hence, we have that

$$\begin{aligned} & \frac{3}{4}\mathbb{E}\left(\sup_{s \in [t', T]} e^{\beta s} |\Delta Y(s)|^2\right) \leq C_1\mathbb{E}(e^{\beta T} |\Delta h|^2) + \frac{3}{a}C_1\mathbb{E}\int_{t'}^T e^{\beta r} |\Delta F(r)|^2 dr \\ & + \frac{3TK e^{\beta \delta}}{a}\left(1 + \frac{144}{1 - 6L^2/a}\right)\mathbb{E}\left(\sup_{r \in [0, T]} e^{\beta r} |\Delta U(r)|^2\right) \\ & + \frac{3K e^{\beta \delta}}{a}\left(1 + \frac{144}{1 - 6L^2/a}\right)\mathbb{E}\int_0^T e^{\beta r} |\Delta V(r)|^2 dr. \end{aligned} \tag{18}$$

Exploiting thus assumptions (A_3) and (A_5) together with the fact that $X^{\cdot, \phi}$ is continuous and bounded, we have that

$$C_1\mathbb{E}(e^{\beta T} |\Delta h|^2) + \frac{3}{a}C_1\mathbb{E}\int_{t'}^T e^{\beta r} |\Delta F(r)|^2 dr \rightarrow 0 \quad \text{as } t' \rightarrow t.$$

Since $(U, V) \in \mathcal{A} \times \mathcal{B}$, and therefore we have that

$$\mathbb{E}\left(\sup_{r \in [0, T]} e^{\beta r} |\Delta U(r)|^2\right) \rightarrow 0,$$

and

$$\mathbb{E}\int_0^T e^{\beta r} |\Delta V(r)|^2 dr \rightarrow 0,$$

as $t' \rightarrow t$, we have that

$$\mathbb{E}\left(\sup_{s \in [t', T]} e^{\beta s} |\Delta Y(s)|^2\right) \rightarrow 0 \quad \text{and} \quad \mathbb{E}\int_{t'}^T e^{\beta r} |\Delta Z(r)|^2 dr \rightarrow 0, \quad \text{as } t' \rightarrow t. \tag{19}$$

We are now left to show that the term $\mathbb{E}\left(\sup_{s \in [t, t']} |Y^t(t) - Y^s(s)|^2\right)$ is also converging to 0 as $t' \rightarrow t$.

Since the map $t \mapsto Y^t(t)$ is deterministic, we have from equation (13),

$$\begin{aligned}
Y^t(t) - Y^s(s) &= \mathbb{E}[Y^t(t) - Y^s(s)] \\
&= \mathbb{E}[h(X^{t,\phi}) - h(X^{s,\phi})] + \mathbb{E} \int_t^T F(r, X^{t,\phi}, Y^t(r), Z^t(r), U_r^t, V_r^t) dr \\
&\quad - \mathbb{E} \int_s^T F(r, X^{s,\phi}, Y^s(r), Z^s(r), U_r^s, V_r^s) dr \\
&= \mathbb{E}[h(X^{t,\phi}) - h(X^{s,\phi})] + \mathbb{E} \int_t^s F(r, X^{t,\phi}, Y^t(r), Z^t(r), U_r^t, V_r^t) dr \\
&\quad + \mathbb{E} \int_s^T [F(r, X^{t,\phi}, Y^t(r), Z^t(r), U_r^t, V_r^t) - F(r, X^{s,\phi}, Y^s(r), Z^s(r), U_r^s, V_r^s)] dr.
\end{aligned}$$

Using then assumption (11) we have

$$\begin{aligned}
&|Y^{t,\phi}(t) - Y^{s,\phi}(s)| \\
&\leq \mathbb{E}|h(X^{t,\phi}) - h(X^{s,\phi})| + \mathbb{E} \int_t^s L(|Y^t(r)| + |Z^t(r)|) dr \\
&\quad + \sqrt{K \int_t^s \mathbb{E} \left(\int_{-\delta}^0 |U^t(r+\theta)|^2 + |V^t(r+\theta)|^2 \alpha(d\theta) \right) dr} \cdot \sqrt{s-t} \\
&\quad + \mathbb{E} \int_t^s |F(r, X^{t,\phi}, 0, 0, 0, 0)| dr \\
&\quad + \mathbb{E} \int_s^T |F(r, X^{t,\phi}, Y^t(r), Z^t(r), U_r^t, V_r^t) - F(r, X^{s,\phi}, Y^t(r), Z^t(r), U_r^t, V_r^t)| dr \\
&\quad + \mathbb{E} \int_s^T L(|Y^t(r) - Y^s(r)| + |Z^t(r) - Z^s(r)|) dr \\
&\quad + \sqrt{K(T-s) \int_s^T \mathbb{E} \left(\int_{-\delta}^0 |U^t(r+\theta) - U^s(r+\theta)|^2 + |V^t(r+\theta) - V^s(r+\theta)|^2 \alpha(d\theta) \right) dr}
\end{aligned}$$

and therefore we obtain

$$\begin{aligned}
& |Y^t(t) - Y^s(s)| \\
& \leq \mathbb{E}|h(X^{t,\phi}) - h(X^{s,\phi})| + L\sqrt{s-t}\sqrt{T\mathbb{E}\sup_{r\in[0,T]}|Y^t(r)|^2 + \mathbb{E}\int_0^T|Z^t(r)|^2dr} \\
& \quad + \sqrt{K}\sqrt{s-t}\sqrt{T\mathbb{E}\sup_{r\in[0,T]}|U^t(r)|^2 + \mathbb{E}\int_0^T|V^t(r)|^2dr} \\
& \quad + (s-t)M(1 + \mathbb{E}\|X^{t,\phi}\|_T^p) \\
& \quad + \mathbb{E}\int_s^T|F(r, X^{t,\phi}, Y^t(r), Z^t(r), U_r^t, V_r^t) - F(r, X^{s,\phi}, Y^t(r), Z^t(r), U_r^t, V_r^t)|dr \\
& \quad + L\sqrt{T-s}\sqrt{T\mathbb{E}\sup_{r\in[0,T]}|Y^t(r) - Y^s(r)|^2 + \mathbb{E}\int_0^T|Z^t(r) - Z^s(r)|^2dr} \\
& \quad + \sqrt{K}\sqrt{T-s}\sqrt{T\mathbb{E}\sup_{r\in[0,T]}|U^t(r) - U^s(r)|^2 + \mathbb{E}\int_0^T|V^t(r) - V^s(r)|^2dr}.
\end{aligned}$$

Taking again into account the fact that $(U, V) \in \mathcal{A} \times \mathcal{B}$, properties (10) and assumptions (A_3) and (A_5) , we infer that

$$\mathbb{E}\left(\sup_{s\in[t,t']}|Y^t(t) - Y^s(s)|\right) \rightarrow 0, \quad \text{as } t' \rightarrow t. \quad (20)$$

Concerning the term $\mathbb{E}\int_0^T|Z^t(r) - Z^{t'}(r)|^2dr$, we see that

$$\begin{aligned}
& \mathbb{E}\int_0^T|Z^t(r) - Z^{t'}(r)|^2dr \\
& = \mathbb{E}\int_0^{t'}|Z^t(r) - Z^{t'}(r)|^2dr + \mathbb{E}\int_{t'}^T|Z^t(r) - Z^{t'}(r)|^2dr \\
& = \mathbb{E}\int_t^{t'}|Z^t(r)|^2dr + \mathbb{E}\int_{t'}^T|Z^t(r) - Z^{t'}(r)|^2dr,
\end{aligned}$$

hence, by (17),

$$\mathbb{E}\int_0^T|Z^t(r) - Z^{t'}(r)|^2dr \rightarrow 0, \quad \text{as } t' \rightarrow t. \quad (21)$$

Step II.

We are now to prove that Γ is a contraction on the space $\mathcal{A} \times \mathcal{B}$ with respect to the norms

$$|||(Y, Z)|||_{\mathcal{A} \times \mathcal{B}} := (|||Y|||_1^2 + |||Z|||_2^2)^{1/2},$$

where

$$\begin{aligned}
|||Y|||_1^2 &:= \sup_{t\in[0,T]} \mathbb{E}\left(\sup_{r\in[0,T]} e^{\beta r} |Y^t(r)|^2\right) \\
|||Z|||_2^2 &:= \sup_{t\in[0,T]} \mathbb{E}\int_0^T e^{\beta r} |Z^t(r)|^2dr.
\end{aligned}$$

Let us recall that $\Gamma : \mathcal{A} \times \mathcal{B} \rightarrow \mathcal{A} \times \mathcal{B}$ is defined by $\Gamma(U, V) = (Y, Z)$, where (Y, Z) is the solution of the BSDE (13).

Let us consider $(U^1, V^1), (U^2, V^2) \in \mathcal{A} \times \mathcal{B}$ and $(Y^1, Z^1) := \Gamma(U^1, V^1)$, $(Y^2, Z^2) := \Gamma(U^2, V^2)$. For the sake of brevity, we will denote in what follows

$$\begin{aligned}\Delta F^t(r) &:= F(r, X^{t,\phi}, Y^{1,t}(r), Z^{1,t}(r), U_r^{1,t}, V_r^{1,t}) \\ &\quad - F(r, X^{t,\phi}, Y^{2,t}(r), Z^{2,t}(r), U_r^{2,t}, V_r^{2,t}), \\ \Delta U^t(r) &:= U^{1,t}(r) - U^{2,t}(r), \quad \Delta V^t(r) := V^{1,t}(r) - V^{2,t}(r), \\ \Delta Y^t(r) &:= Y^{1,t}(r) - Y^{2,t}(r), \quad \Delta Z^t(r) := Z^{1,t}(r) - Z^{2,t}(r).\end{aligned}$$

Proceeding as in **Step I**, we have from Itô's formula, for any $s \in [t, T]$ and $\beta > 0$,

$$\begin{aligned}& e^{\beta s} |\Delta^t Y(s)|^2 + \beta \int_s^T e^{\beta r} |\Delta Y^t(r)|^2 dr + \int_s^T e^{\beta r} |\Delta Z^t(r)|^2 dr \\ &= 2 \int_s^T e^{\beta r} \langle \Delta Y^t(r), \Delta F^t(r) \rangle dr - 2 \int_s^T e^{\beta r} \langle \Delta Y^t(r), \Delta Z^t(r) \rangle dW(r).\end{aligned}\tag{22}$$

Noticing that it holds

$$\begin{aligned}& \frac{2K}{a} \int_s^T e^{\beta r} \left(\int_{-\delta}^0 (|\Delta U^t(r+\theta)|^2 + |\Delta V^t(r+\theta)|^2) \alpha(d\theta) \right) dr \\ & \leq \frac{2K}{a} \int_{-\delta}^0 \left(\int_s^T e^{\beta r} (|\Delta U^t(r+\theta)|^2 + |\Delta V^t(r+\theta)|^2) dr \right) \alpha(d\theta) \\ & \leq \frac{2K}{a} \int_{-\delta}^0 \left(\int_{s+r}^{T+r} e^{\beta(r'-\theta)} (|\Delta U^t(r')|^2 + |\Delta V^t(r')|^2) dr' \right) \alpha(d\theta) \\ & \leq \frac{2K}{a} \int_{-\delta}^0 e^{-\beta\theta} \alpha(d\theta) \cdot \int_{s-\delta}^T e^{\beta r} (|\Delta U^t(r)|^2 + |\Delta V^t(r)|^2) dr \\ & \leq \frac{2K e^{\beta\delta}}{a} \int_{s-\delta}^T e^{\beta r} (|\Delta U^t(r)|^2 + |\Delta V^t(r)|^2) dr.\end{aligned}$$

we immediately have, from assumptions (A₃)–(A₅), that for any $a > 0$,

$$\begin{aligned}
& 2 \left| \int_s^T e^{\beta r} \langle \Delta Y^t(r), \Delta F^t(r) \rangle dr \right| \leq 2 \int_s^T e^{\beta r} |\langle \Delta Y^t(r), \Delta F^t(r) \rangle| dr \\
& \leq a \int_s^T e^{\beta r} |\Delta Y^t(r)|^2 + \frac{1}{a} \int_s^T e^{\beta r} |\Delta F^t(r)|^2 dr \\
& \leq a \int_s^T e^{\beta r} |\Delta Y^t(r)|^2 + \frac{2}{a} \int_s^T e^{\beta r} L^2 (|\Delta Y^t(r)| + |\Delta Z^t(r)|)^2 dr \\
& \quad + \frac{2}{a} \int_s^T e^{\beta r} \left(K \int_{-\delta}^0 (|\Delta U^t(r+\theta)|^2 + |\Delta V^t(r+\theta)|^2) \alpha(d\theta) \right) dr \\
& \leq a \int_s^T e^{\beta r} |\Delta Y^t(r)|^2 + \frac{4L^2}{a} \int_s^T e^{\beta r} (|\Delta Y^t(r)|^2 + |\Delta Z^t(r)|^2) dr \\
& \quad + \frac{2Ke^{\beta\delta}}{a} \int_{s-\delta}^T e^{\beta r} (|\Delta U^t(r)|^2 + |\Delta V^t(r)|^2) dr.
\end{aligned} \tag{23}$$

Therefore equation (22) yields

$$\begin{aligned}
& e^{\beta s} |\Delta Y^t(s)|^2 + \left(\beta - a - \frac{4L^2}{a} \right) \int_s^T e^{\beta r} |\Delta Y^t(r)|^2 dr + \left(1 - \frac{4L^2}{a} \right) \int_s^T e^{\beta r} |\Delta Z^t(r)|^2 dr \\
& \leq \frac{2Ke^{\beta\delta}}{a} T \sup_{r \in [0, T]} e^{\beta r} |\Delta U^t(r)|^2 + \frac{2Ke^{\beta\delta}}{a} \int_0^T e^{\beta r} |\Delta V^t(r)|^2 dr \\
& \quad - 2 \int_s^T e^{\beta r} \langle \Delta Y^t(r), \Delta Z^t(r) \rangle dW(r).
\end{aligned} \tag{24}$$

Let now $\beta, a > 0$ satisfying

$$\beta > a + \frac{4L^2}{a} \quad \text{and} \quad 1 > \frac{4L^2}{a}, \tag{25}$$

we have that

$$\begin{aligned}
& \left(1 - \frac{4L^2}{a} \right) \mathbb{E} \int_s^T e^{\beta r} |\Delta Z^t(r)|^2 dr \\
& \leq \frac{2TKe^{\beta\delta}}{a} \mathbb{E} \left(\sup_{r \in [0, T]} e^{\beta r} |\Delta U^t(r)|^2 \right) + \frac{2Ke^{\beta\delta}}{a} \mathbb{E} \int_0^T e^{\beta r} |\Delta V^t(r)|^2 dr.
\end{aligned} \tag{26}$$

Exploiting now Burkholder–Davis–Gundy’s inequality, we have that

$$\begin{aligned}
& 2 \mathbb{E} \left[\sup_{s \in [t, T]} \left| \int_s^T e^{\beta r} \langle \Delta Y^t(r), \Delta Z^t(r) \rangle dW(r) \right| \right] \\
& \leq 4 \mathbb{E} \left[\sup_{s \in [t, T]} \left| \int_t^s e^{\beta r} \langle \Delta Y^t(r), \Delta Z^t(r) \rangle dW(r) \right| \right] \\
& \leq \frac{1}{2} \mathbb{E} \left(\sup_{s \in [t, T]} e^{\beta s} |\Delta Y^t(s)|^2 \right) + 72 \mathbb{E} \int_t^T e^{\beta r} |\Delta Z^t(r)|^2 dr,
\end{aligned}$$

which implies

$$\begin{aligned}
& \mathbb{E} \left(\sup_{s \in [t, T]} e^{\beta s} |\Delta Y^t(s)|^2 \right) \\
& \leq \frac{2K e^{\beta \delta}}{a} T \mathbb{E} \left(\sup_{s \in [0, T]} e^{\beta s} |\Delta U^t(s)|^2 \right) + \frac{2K e^{\beta \delta}}{a} \mathbb{E} \int_0^T e^{\beta r} |\Delta V^t(r)|^2 dr \\
& \quad + 2 \mathbb{E} \left[\sup_{s \in [t, T]} \left| \int_s^T e^{\beta r} \langle \Delta Y^t(r), \Delta Z^t(r) \rangle dW(r) \right| \right] \\
& \leq \frac{2K e^{\beta \delta}}{a} T \mathbb{E} \left(\sup_{s \in [0, T]} e^{\beta s} |\Delta U^t(s)|^2 \right) + \frac{2K e^{\beta \delta}}{a} \mathbb{E} \int_0^T e^{\beta r} |\Delta V^t(r)|^2 dr \\
& \quad + \frac{1}{2} \mathbb{E} \left(\sup_{s \in [t, T]} e^{\beta s} |\Delta Y^t(s)|^2 \right) + 72 \mathbb{E} \int_t^T e^{\beta r} |\Delta Z^t(r)|^2 dr.
\end{aligned}$$

Hence, we have that

$$\begin{aligned}
& \mathbb{E} \left(\sup_{s \in [t, T]} e^{\beta s} |\Delta Y^t(s)|^2 \right) \\
& \leq \frac{4TK e^{\beta \delta}}{a} C_1 \mathbb{E} \left(\sup_{s \in [0, T]} e^{\beta s} |\Delta U^t(s)|^2 \right) + \frac{4K e^{\beta \delta}}{a} C_1 \mathbb{E} \int_0^T e^{\beta r} |\Delta V^t(r)|^2 dr,
\end{aligned} \tag{27}$$

where we have denoted by $C_1 := 1 + \frac{72}{1-4L^2/a}$.

Let us now consider the term $\mathbb{E} \left(\sup_{s \in [0, t]} e^{\beta s} |\Delta Y(s)|^2 \right)$. From equation (13), we see that,

$$\begin{aligned}
& \mathbb{E} \left(\sup_{s \in [0, t]} e^{\beta s} |\Delta Y^t(s)|^2 \right) = \mathbb{E} \left(\sup_{s \in [0, t]} e^{\beta s} |Y^{1,t}(s) - Y^{2,t}(s)|^2 \right) \\
& = \mathbb{E} \left(\sup_{s \in [0, t]} e^{\beta s} |Y^{1,s}(s) - Y^{2,s}(s)|^2 \right) = \sup_{s \in [0, t]} e^{\beta s} |\Delta Y^s(s)|^2 = \sup_{s \in [0, t]} \mathbb{E} (e^{\beta s} |\Delta Y^s(s)|^2)
\end{aligned} \tag{28}$$

so that, exploiting Itô's formula and proceeding as above, we obtain that

$$\begin{aligned}
& \mathbb{E} (e^{\beta s} |\Delta Y^s(s)|^2) \\
& \leq \frac{2TK e^{\beta \delta}}{a} \mathbb{E} \left(\sup_{r \in [0, T]} e^{\beta r} |\Delta U^s(r)|^2 \right) + \frac{2K e^{\beta \delta}}{a} \mathbb{E} \int_0^T e^{\beta r} |\Delta V^s(r)|^2 dr.
\end{aligned} \tag{29}$$

Thus from inequalities (26–29) we obtain

$$\begin{aligned}
& \mathbb{E} \left(\sup_{s \in [0, T]} e^{\beta r} |\Delta Y^t(s)|^2 \right) + \mathbb{E} \int_0^T e^{\beta r} |\Delta Z^t(r)|^2 dr \\
& \leq \frac{4TK e^{\beta \delta}}{a} C_1 \mathbb{E} \left(\sup_{s \in [0, T]} e^{\beta s} |\Delta U^t(s)|^2 \right) + \frac{4K e^{\beta \delta}}{a} C_1 \mathbb{E} \int_0^T e^{\beta r} |\Delta V^t(r)|^2 dr \\
& \quad + \frac{2TK e^{\beta \delta}}{a(1-4L^2/a)} \mathbb{E} \left(\sup_{r \in [0, T]} e^{\beta r} |\Delta U^t(r)|^2 \right) + \frac{2K e^{\beta \delta}}{a(1-4L^2/a)} \mathbb{E} \int_0^T e^{\beta r} |\Delta V^t(r)|^2 dr \\
& \quad + \frac{2TK e^{\beta \delta}}{a} \cdot \sup_{s \in [0, t]} \mathbb{E} \left(\sup_{r \in [0, T]} e^{\beta r} |\Delta U^s(r)|^2 \right) + \frac{2K e^{\beta \delta}}{a} \cdot \sup_{s \in [0, t]} \mathbb{E} \int_0^T e^{\beta r} |\Delta V^s(r)|^2 dr.
\end{aligned}$$

Passing then to the supremum for $t \in [0, T]$ we get

$$\begin{aligned} & |||Y^1 - Y^2|||_1^2 + |||Z^1 - Z^2|||_2^2 \\ & \leq \frac{2Ke^{\beta\delta}}{a} \left(3 + \frac{145}{1 - 4L^2/a} \right) \max\{1, T\} \left[|||U^1 - U^2|||_1^2 + |||V^1 - V^2|||_2^2 \right]. \end{aligned}$$

By choosing now $a := \frac{4L^2}{\gamma}$ and β slightly bigger than $\gamma + \frac{4L^2}{\gamma}$, condition (25) is satisfied and, by (12) we have that

$$\frac{2Ke^{\beta\delta}}{a} \left(3 + \frac{145}{1 - 4L^2/a} \right) \max\{1, T\} < 1. \quad (30)$$

Eventually, since U and V were chosen arbitrarily, it follows that the application Γ is a contraction on the space $\mathcal{A} \times \mathcal{B}$. Therefore there exists a unique fixed point $\Gamma(Y, Z) = (Y, Z) \in \mathcal{A} \times \mathcal{B}$ and this finishes the proof of the existence and a uniqueness of a solution to equation (8). \blacksquare

Remark 9 Using Itô's formula and proceeding as in the proof of Theorem 7, we can easily show that the solution $(Y^{t,\phi}, Z^{t,\phi})$ to equation (8) satisfies the following inequality. For any $q \geq 1$, there exists $C > 0$ such that

$$\begin{aligned} & \mathbb{E} \left(\sup_{r \in [0, T]} |Y^{t,\phi}(r)|^{2q} \right) + \mathbb{E} \left(\int_0^T |Z^{t,\phi}(r)|^2 dr \right)^q \\ & \leq C \left[\mathbb{E} |h(X^{t,\phi})|^{2q} + \mathbb{E} \left(\int_0^T |F(r, X^{t,\phi}, 0, 0, 0, 0)|^2 dr \right)^{2q} \right] \\ & \leq C(1 + \|\phi\|_T^{2q}). \end{aligned} \quad (31)$$

4 Path-dependent PDE – proof of the existence theorem

The current section is devoted to the study of viscosity solution to the path-dependent equation (1). In particular, in order to obtain existence of a viscosity solution, we will impose some additional assumptions on the generator f and on the terminal condition h in equation (8), in particular we will assume that the generator f depends only on past values assumed by Y and not by past values assumed by the control Z . In what follows we will assume the following to hold.

Let

$$f : [0, T] \times \mathbb{A} \times \mathbb{R} \times \mathbb{R}^d \times \mathcal{C}([-\delta, 0]; \mathbb{R}) \rightarrow \mathbb{R},$$

and

$$h : \mathbb{A} \rightarrow \mathbb{R},$$

such that the following holds.

(A₆) the functions f and h are continuous; also $f(\cdot, \cdot, y, z, \hat{y})$ is non-anticipative;

(A₇) there exist $L, K, M > 0$ and $p \geq 1$ such that for any $(t, \phi) \in [0, T] \times \mathbb{A}$, $y, y' \in \mathbb{R}$, $z, z' \in \mathbb{R}^d$ and $\hat{y}, \hat{y}' \in \mathcal{C}([-\delta, 0]; \mathbb{R})$:

- (i) $|f(t, \phi, y, z, \hat{y}) - f(t, \phi, y', z', \hat{y})| \leq L(|y - y'| + |z - z'|),$
- (ii) $|f(t, \phi, y, z, \hat{y}) - f(t, \phi, y, z, \hat{y}')|^2 \leq K \int_{-\delta}^0 |\hat{y}(\theta) - \hat{y}'(\theta)|^2 \alpha(d\theta),$
- (iii) $|f(t, \phi, 0, 0, 0)| \leq M(1 + \|\phi\|_T^p),$
- (iv) $|h(\phi)| \leq M(1 + \|\phi\|_T^p),$

being α a probability measure on $([-\delta, 0], \mathcal{B}([-\delta, 0]))$.

Remark 10 As example, the next generators f satisfy assumptions (A₆), (A₇):

$$\begin{aligned} f_1(t, \phi, y, z, \hat{y}) &:= K \int_{-\delta}^0 \hat{y}(s) ds, \\ f_2(t, \phi, y, z, \hat{y}) &:= K \hat{y}(t - \delta). \end{aligned}$$

In general, being $g : [0, T] \rightarrow \mathbb{R}$ a measurable and bounded function with $g(t) = 0$ for $t < 0$, the following linear time delayed generator

$$f(t, \phi, y, z, \hat{y}) = \int_{-\delta}^0 g(t + \theta) \hat{y}(\theta) \alpha(d\theta),$$

satisfies assumptions (A₆), (A₇).

Next is the main result of the present paper.

Theorem 11 (Existence) Let us assume that assumptions (A₁), (A₂), (A₆), (A₇) hold. If the delay δ or the Lipschitz constant K are sufficiently small, i.e. condition (12) is verified, then the path-dependent PDE (1) admits at least one viscosity solution.

The proof of this result uses in an essential way the nonlinear representation Feynman-Kac type formula, which links the functional SDE (9) to a suitable BSDE with time-delayed generators.

Under assumptions (A₁), (A₂), (A₆), (A₇), it follows from Theorem 7, in the case $m = 1$, that, for each $(t, \phi) \in [0, T] \times \mathbb{A}$, there exists a unique solution $(X^{t, \phi}, Y^{t, \phi}, Z^{t, \phi})$ of \mathbb{G}^t -progressively measurable processes such that $(Y^{t, \phi}, Z^{t, \phi}) \in \mathcal{S}_t^{2,1} \times \mathcal{H}_t^{2,1 \times d'}$, with $Y^{t, \phi}(s) = Y^{s, \phi}(s)$, for any $s \in [0, T]$, solution to the BSDE:

$$Y^{t, \phi}(s) = h(X^{t, \phi}) + \int_s^T f(r, X^{t, \phi}, Y^{t, \phi}(r), Z^{t, \phi}(r), Y_r^{t, \phi}) dr - \int_s^T Z^{t, \phi}(r) dW(r), \quad (32)$$

for all $s \in [t, T]$.

Let us further observe that the generator f depends on ω only via the forward process $X^{t, x}$.

Before proving Theorem 11, we need to prove some results. In particular, let us define the function $u : [0, T] \times \mathbb{A} \rightarrow \mathbb{R}$ by

$$u(t, \phi) := Y^{t, \phi}(t), \quad (t, \phi) \in [0, T] \times \mathbb{A}, \quad (33)$$

notice that $u(t, \phi)$ is a deterministic function since $Y^{t, \phi}(t)$ is $\mathcal{G}_t^t \equiv \mathcal{N}$ -measurable.

Theorem 12 *Under the assumptions of Theorem 11, the function u is continuous.*

Proof. Let us first prove the continuity of $\mathbb{A} \ni \phi \mapsto u(t, \phi)$, uniformly with respect to $t \in [0, T]$.

Let us thus take $t \in [0, T]$, $\phi, \phi' \in \mathbb{A}$, and let us denote

$$\Delta Y(r) := Y^{t, \phi}(r) - Y^{t, \phi'}(r), \quad \Delta Z(r) := Z^{t, \phi}(r) - Z^{t, \phi'}(r)$$

and

$$\Delta f(r) := f(r, X^{t, \phi}, Y^{t, \phi}(r), Z^{t, \phi}(r), Y_r^{t, \phi}) - f(r, X^{t, \phi}, Y^{t, \phi}(r), Z^{t, \phi}(r), Y_r^{t, \phi'}).$$

By Itô's formula we have, for any $\beta > 0$ and any $s \in [t, T]$,

$$\begin{aligned} e^{\beta s} |\Delta Y(s)|^2 + \beta \int_s^T e^{\beta r} |\Delta Y(r)|^2 dr + \int_s^T e^{\beta r} |\Delta Z(r)|^2 dr &= e^{\beta T} |\Delta Y(T)|^2 \\ &+ 2 \int_s^T e^{\beta r} \langle \Delta Y(r), f(r, X^{t, \phi}, Y^{t, \phi}(r), Z^{t, \phi}(r), Y_r^{t, \phi}) \\ &\quad - f(r, X^{t, \phi'}, Y^{t, \phi'}(r), Z^{t, \phi'}(r), Y_r^{t, \phi'}) \rangle dr \\ &- 2 \int_s^T e^{\beta r} \langle \Delta Y(r), \Delta Z(r) \rangle dW(r). \end{aligned}$$

Exploiting thus assumptions $(A_1), (A_2), (A_6), (A_7)$, we have for any $a > 0$,

$$\begin{aligned} 2 \left| \int_s^T e^{\beta r} \langle \Delta Y(r), f(r, X^{t, \phi}, Y^{t, \phi}(r), Z^{t, \phi}(r), Y_r^{t, \phi}) \right. \\ \left. - f(r, X^{t, \phi'}, Y^{t, \phi'}(r), Z^{t, \phi'}(r), Y_r^{t, \phi'}) \rangle dr \right| \leq \\ \leq a \int_s^T e^{\beta r} |\Delta Y(r)|^2 + \frac{3}{a} \int_s^T e^{\beta r} |\Delta f(r)|^2 dr + \frac{3}{a} \int_s^T e^{\beta r} L^2 (|\Delta Y(r)| + |\Delta Z(r)|)^2 dr \\ + \frac{3Ke^{\beta \delta}}{a} \int_{s-\delta}^T e^{\beta r} |\Delta Y(r)|^2 dr, \end{aligned}$$

so that it holds

$$\begin{aligned} e^{\beta s} |\Delta Y(s)|^2 + \left(\beta - a - \frac{6L^2}{a} \right) \int_s^T e^{\beta r} |\Delta Y(r)|^2 dr + \left(1 - \frac{6L^2}{a} \right) \int_s^T e^{\beta r} |\Delta Z(r)|^2 dr \\ \leq e^{\beta T} |\Delta Y(T)|^2 + \frac{3}{a} \int_s^T e^{\beta r} |\Delta f(r)|^2 dr + \frac{3Ke^{\beta \delta}}{a} \int_{s-\delta}^T e^{\beta r} |\Delta Y(r)|^2 dr \\ - 2 \int_s^T e^{\beta r} \langle \Delta Y(r), \Delta Z(r) \rangle dW(r). \end{aligned}$$

Let now $\beta, a > 0$ such that

$$\beta > a + \frac{6L^2}{a} \quad \text{and} \quad 1 > \frac{6L^2}{a}, \quad (34)$$

then

$$\begin{aligned} & \left(1 - \frac{6L^2}{a}\right) \mathbb{E} \int_t^T e^{\beta r} |\Delta Z(r)|^2 dr \\ & \leq \mathbb{E}(e^{\beta T} |\Delta Y(T)|^2) + \frac{3}{a} \mathbb{E} \int_t^T e^{\beta r} |\Delta f(r)|^2 dr + \frac{3Ke^{\beta\delta}}{a} \mathbb{E} \int_{t-\delta}^T e^{\beta r} |\Delta Y(r)|^2 dr \end{aligned} \quad (35)$$

and therefore, exploiting Burkholder–Davis–Gundy’s inequality, we have that

$$\begin{aligned} & 2\mathbb{E} \left[\sup_{s \in [t, T]} \left| \int_s^T e^{\beta r} \langle \Delta Y(r), \Delta Z(r) \rangle dW(r) \right| \right] \\ & \leq \frac{1}{4} \mathbb{E} \left(\sup_{s \in [t, T]} e^{\beta s} |\Delta Y(s)|^2 \right) + 144 \mathbb{E} \int_t^T e^{\beta r} |\Delta Z(r)|^2 dr, \end{aligned}$$

which immediately implies

$$\begin{aligned} & \frac{3}{4} \mathbb{E} \left(\sup_{s \in [t, T]} e^{\beta s} |\Delta Y(s)|^2 \right) \\ & \leq \mathbb{E}(e^{\beta T} |\Delta Y(T)|^2) + \frac{3}{a} \mathbb{E} \int_t^T e^{\beta r} |\Delta f(r)|^2 dr + \frac{3Ke^{\beta\delta}}{a} \mathbb{E} \int_{t-\delta}^T e^{\beta r} |\Delta Y(r)|^2 dr \\ & \quad + 144 \mathbb{E} \int_t^T e^{\beta r} |\Delta Z(r)|^2 dr \\ & \leq \mathbb{E}(e^{\beta T} |\Delta Y(T)|^2) + \frac{3}{a} \mathbb{E} \int_t^T e^{\beta r} |\Delta f(r)|^2 dr + \frac{3Ke^{\beta\delta}}{a} \mathbb{E} \int_{t-\delta}^T e^{\beta r} |\Delta Y(r)|^2 dr \\ & \quad + 144 \left(1 - \frac{6L^2}{a}\right)^{-1} \left[\mathbb{E}(e^{\beta T} |\Delta Y(T)|^2) + \frac{3}{a} \mathbb{E} \int_t^T e^{\beta r} |\Delta f(r)|^2 dr \right] \\ & \quad + 144 \left(1 - \frac{6L^2}{a}\right)^{-1} \left[\frac{3Ke^{\beta\delta}}{a} \mathbb{E} \int_{t-\delta}^T e^{\beta r} |\Delta Y(r)|^2 dr \right]. \end{aligned}$$

Hence,

$$\begin{aligned} & \frac{3}{4} \mathbb{E} \left(\sup_{s \in [t, T]} e^{\beta s} |\Delta Y(s)|^2 \right) \\ & \leq C_1 \mathbb{E}(e^{\beta T} |\Delta Y(T)|^2) + \frac{3}{a} C_1 \mathbb{E} \int_t^T e^{\beta r} |\Delta f(r)|^2 dr + \frac{3Ke^{\beta\delta}}{a} C_1 \mathbb{E} \int_{t-\delta}^T e^{\beta r} |\Delta Y(r)|^2 dr, \end{aligned}$$

where we have denoted

$$C_1 := 1 + \frac{144}{1 - 6L^2/a}. \quad (36)$$

Choosing now $a := \frac{6L^2}{\gamma}$ and β slightly bigger than $\gamma + \frac{6L^2}{\gamma}$, condition (34) is satisfied and

$$T \frac{3Ke^{\beta\delta}}{a} C_1 < \frac{1}{4}, \quad (37)$$

by (12). We then have

$$\begin{aligned} & \frac{1}{2} \mathbb{E} \left(\sup_{s \in [t, T]} e^{\beta s} |\Delta Y(s)|^2 \right) \\ & \leq C_1 \mathbb{E} (e^{\beta T} |\Delta Y(T)|^2) + \frac{3C_1}{a} \mathbb{E} \int_t^T e^{\beta r} |\Delta f(r)|^2 dr + \frac{3K\delta e^{\beta\delta}}{a} C_1 \mathbb{E} \left(\sup_{s \in [t-\delta, t]} e^{\beta s} |\Delta Y(s)|^2 \right). \end{aligned} \quad (38)$$

Exploiting the initial conditions satisfied by the $Y^{t, \phi}$, we can rewrite equation (38) as

$$\begin{aligned} & \mathbb{E} \left(\sup_{s \in [t-\delta, t]} e^{\beta s} |\Delta Y(s)|^2 \right) = \mathbb{E} \left(\sup_{s \in [t-\delta, t]} e^{\beta s} |Y^{t, \phi}(s) - Y^{t, \phi'}(s)|^2 \right) \\ & = \sup_{s \in [t-\delta, t]} e^{\beta s} |Y^{s, \phi}(s) - Y^{s, \phi'}(s)|^2 \leq \sup_{s \in [t-\delta, t]} \mathbb{E} \left(\sup_{r \in [s, T]} e^{\beta(s-r)} e^{\beta r} |Y^{s, \phi}(r) - Y^{s, \phi'}(r)|^2 \right), \end{aligned}$$

and therefore we obtain that

$$\begin{aligned} & \mathbb{E} \left(\sup_{s \in [t, T]} e^{\beta s} |\Delta Y(s)|^2 \right) \leq 2C_1 \mathbb{E} (e^{\beta T} |\Delta Y(T)|^2) \\ & + \frac{6C_1}{a} \mathbb{E} \int_t^T e^{\beta r} |\Delta f(r)|^2 dr \\ & + \frac{6K\delta e^{\beta\delta}}{a} C_1 \sup_{s \in [t-\delta, t]} \mathbb{E} \left(\sup_{r \in [s, T]} e^{\beta(s-r)} e^{\beta r} |Y^{s, \phi}(r) - Y^{s, \phi'}(r)|^2 \right). \end{aligned}$$

Passing to the supremum for $t \in [0, T]$ we have that,

$$\begin{aligned} & \sup_{t \in [0, T]} \mathbb{E} \left(\sup_{s \in [t, T]} e^{\beta s} |Y^{t, \phi}(s) - Y^{t, \phi'}(s)|^2 \right) \\ & \leq 2C_1 \sup_{t \in [0, T]} \mathbb{E} (e^{\beta T} |h(X^{t, \phi}) - h(X^{t, \phi'})|^2) + \frac{6C_1}{a} \sup_{t \in [0, T]} \mathbb{E} \int_t^T e^{\beta r} |\Delta f(r)|^2 dr \\ & + \frac{6K\delta e^{\beta\delta}}{a} C_1 \sup_{s \in [0, T]} \mathbb{E} \left(\sup_{r \in [s, T]} e^{\beta r} |Y^{s, \phi}(r) - Y^{s, \phi'}(r)|^2 \right). \end{aligned} \quad (39)$$

We can now see that

$$\begin{aligned} & \mathbb{E} \left(\sup_{s \in [0, t]} e^{\beta s} |Y^{t, \phi}(s) - Y^{t, \phi'}(s)|^2 \right) = \sup_{s \in [0, t]} e^{\beta s} |Y^{s, \phi}(s) - Y^{s, \phi'}(s)|^2 \\ & \leq \sup_{s \in [0, T]} \mathbb{E} \left(\sup_{r \in [s, T]} e^{\beta r} |Y^{s, \phi}(r) - Y^{s, \phi'}(r)|^2 \right), \end{aligned} \quad (40)$$

so that we can apply again inequality (39).

From inequalities (39) and (40) we can conclude that

$$\begin{aligned}
& \sup_{t \in [0, T]} \mathbb{E} \left(\sup_{s \in [0, T]} e^{\beta s} |Y^{t, \phi}(s) - Y^{t, \phi'}(s)|^2 \right) \\
& \leq 4C_1 \sup_{t \in [0, T]} \mathbb{E} (e^{\beta T} |h(X^{t, \phi}) - h(X^{t, \phi'})|^2) + \frac{12C_1}{a} \sup_{t \in [0, T]} \mathbb{E} \int_t^T e^{\beta r} |\Delta f(r)|^2 dr \\
& \quad + \frac{12K\delta e^{\beta \delta}}{a} C_1 \sup_{s \in [0, T]} \mathbb{E} \left(\sup_{r \in [s, T]} e^{\beta r} |Y^{s, \phi}(r) - Y^{s, \phi'}(r)|^2 \right).
\end{aligned}$$

Since $\delta \leq T$, by (37) we also have

$$\frac{12K\delta e^{\beta \delta}}{a} C_1 < 1$$

and so

$$\begin{aligned}
& \left(1 - \frac{12K\delta e^{\beta \delta}}{a} C_1 \right) \sup_{t \in [0, T]} \mathbb{E} \left(\sup_{s \in [0, T]} e^{\beta s} |Y^{t, \phi}(s) - Y^{t, \phi'}(s)|^2 \right) \\
& \leq 4C_1 \sup_{t \in [0, T]} \mathbb{E} (e^{\beta T} |h(X^{t, \phi}) - h(X^{t, \phi'})|^2) \\
& \quad + \frac{12C_1}{a} \sup_{t \in [0, T]} \mathbb{E} \int_t^T e^{\beta r} |f(r, X^{t, \phi}, Y^{t, \phi}(r), Z^{t, \phi}(r), Y_r^{t, \phi}) + \\
& \quad - f(r, X^{t, \phi'}, Y^{t, \phi}(r), Z^{t, \phi}(r), Y_r^{t, \phi})|^2 dr.
\end{aligned}$$

Let us now fix $\phi \in \mathbb{A}$. In order to prove that u is continuous in ϕ , uniformly with respect to $t \in [0, T]$, it is enough to show that

$$\begin{aligned}
& \mathbb{E} (|h(X^{t, \phi}) - h(X^{t, \phi'})|^2) \\
& + \mathbb{E} \int_0^T |f(r, X^{t, \phi}, Y^{t, \phi}(r), Z^{t, \phi}(r), Y_r^{t, \phi}) - f(r, X^{t, \phi'}, Y^{t, \phi}(r), Z^{t, \phi}(r), Y_r^{t, \phi})|^2 dr
\end{aligned}$$

converge to 0 as $\phi' \rightarrow \phi$, uniformly in $t \in [0, T]$.

Since we have no guarantee that the family $\{|Z^{t, \phi}|^2\}_{t \in [0, T]}$ is uniformly integrable, we will use the Lipschitz property of f in the argument (y, z, u) in order to replace $[0, T]$ with a finite subset. By Theorem 7, the mapping $t \mapsto (Y^{t, \phi}, Z^{t, \phi})$ is continuous from $[0, T]$ into $\mathcal{S}_0^{2,1} \times \mathcal{H}_0^{2,d'}$ and therefore uniformly continuous. Consequently, as $n \rightarrow \infty$, we have that

$$\sup_{|t-t'| \leq \frac{1}{n}} \mathbb{E} \left[\sup_{s \in [0, T]} (Y^{t, \phi}(s) - Y^{t', \phi}(s))^2 + \int_0^T (Z^{t, \phi}(s) - Z^{t', \phi}(s))^2 ds \right] \rightarrow 0.$$

Let, for $n \in \mathbb{N}^*$, $\pi_n := \{0, \frac{T}{n}, \dots, \frac{(n-1)T}{n}, T\}$, then, by (A₆), we see that

$$\begin{aligned}
& \sup_{t \in [0, T]} \sup_{t' \in \pi_n} \mathbb{E} \int_0^T |f(r, X^{t, \phi}, Y^{t', \phi}(r), Z^{t', \phi}(r), Y_r^{t', \phi}) + \\
& \quad - f(r, X^{t, \phi'}, Y^{t', \phi}(r), Z^{t', \phi}(r), Y_r^{t', \phi})|^2 dr,
\end{aligned}$$

converges to

$$\sup_{t \in [0, T]} \mathbb{E} \int_0^T |f(r, X^{t, \phi}, Y^{t, \phi}(r), Z^{t, \phi}(r), Y_r^{t, \phi}) - f(r, X^{t, \phi'}, Y^{t, \phi'}(r), Z^{t, \phi'}(r), Y_r^{t, \phi'})|^2 dr,$$

uniformly in ϕ' . We are thus left to prove that

$$\begin{aligned} & \mathbb{E}(|h(X^{t, \phi}) - h(X^{t, \phi'})|^2) \\ & + \mathbb{E} \int_0^T |f(r, X^{t, \phi}, Y^{t', \phi}(r), Z^{t', \phi}(r), Y_r^{t', \phi}) - f(r, X^{t, \phi'}, Y^{t', \phi'}(r), Z^{t', \phi'}(r), Y_r^{t', \phi'})|^2 dr \end{aligned}$$

converge to 0 as $\phi' \rightarrow \phi$, uniformly in $t \in [0, T]$, for fixed $n \in \mathbb{N}^*$ and $t' \in \pi_n$.

Let us thus introduce the modulus of continuity of the functions h and f :

$$\begin{aligned} m_{h, f}(\epsilon, \mathcal{K}, \mathcal{U}, \kappa) &= \\ &:= \sup_{\substack{\phi', \phi'' \in \mathcal{K}, t \in [0, T], (y, u) \in \mathcal{U} \\ |z| \leq \kappa, \|\phi - \phi'\|_T \leq \epsilon}} (|h(\phi') - h(\phi'')| + |f(t, \phi', y, z, u) - f(t, \phi'', y, z, u)|), \end{aligned}$$

where $\epsilon > 0$, \mathcal{K} is a compact in \mathbb{A} , \mathcal{U} is a compact in $\mathbb{R} \times L^2([-\delta, 0]; \mathbb{R})$ and $\kappa \in \mathbb{R}_+$.

Let $\epsilon > 0$ be fixed, but arbitrary; with no loss generality, we can suppose that the function ϕ' lies in a compact $\mathcal{K} \subseteq \mathbb{A}$.

By Theorem 4, we know that the family $\{(X^{t, \phi'}, Y^{t', \phi'})\}_{(t, \phi') \in [0, T] \times \mathcal{K}}$ is tight with respect to the product topology on $\mathbb{A} \times \mathcal{C}([0, T])$, and therefore, for every $\epsilon > 0$, there exist compact subsets $\mathcal{K}_\epsilon \subseteq \mathbb{A}$ and $\mathcal{K}'_\epsilon \subseteq \mathcal{C}([0, T])$ such that

$$\mathbb{P}(X^{t, \phi'} \in \mathcal{K}_\epsilon, Y^{t', \phi'} \in \mathcal{K}'_\epsilon) \geq 1 - \epsilon, \quad \text{for all } (t, \phi') \in [0, T] \times \mathcal{K}.$$

For ease of the notation, let us define $\Phi : \mathbb{A} \times \mathbb{A} \times \mathcal{C}([0, T]) \times \mathbb{R}^d \rightarrow \mathbb{R}$ by

$$\Phi(r, \phi', \phi'', y, z) := \frac{1}{T} |h(\phi') - h(\phi'')|^2 + |f(r, \phi', y(r), z, y_r) - f(r, \phi'', y(r), z, y_r)|^2.$$

We can see by (A_6) , that it holds

$$\Phi(\phi', \phi'', y, z) \leq C \left(1 + \|\phi'\|_T^{2p} + \|\phi''\|_T^{2p} + \|y\|_T^2 + |z|^2 \right),$$

where in what follows we will denote by C several possibly different constants depending only on K, L, M and T . Then, for all $t \in [0, T]$, $\phi', \phi'' \in \mathbb{A}$, we have from the a priori estimate (31) on the processes $Y^{t, \phi}$ and $Z^{t, \phi}$, we have that,

$$\mathbb{E} \left[\int_0^T \Phi(r, X^{t, \phi'}, X^{t, \phi''}, Y^{t', \phi}, Z^{t', \phi}(r)) dr \right]^{p'} \leq C \left(1 + \|\phi'\|_T^{pp'} + \|\phi''\|_T^{pp'} \right).$$

Let now \mathcal{U}_ϵ be the image of $[0, T] \times \mathcal{K}'_\epsilon$ through the continuous application

$$(r, y) \mapsto (y(r), y_r),$$

and we also have that, \mathcal{U} is compact in $\mathbb{R} \times L^2([-\delta, 0]; \mathbb{R})$.

For arbitrary $\epsilon', \kappa > 0$, we see that,

$$\begin{aligned}
& \mathbb{E} \int_0^T \Phi \left(r, X^{t,\phi}, X^{t,\phi'}, Y^{t',\phi}, Z^{t',\phi}(r) \right) dr \\
& \leq \mathbb{E} \int_0^T \Phi \left(X^{t,\phi}, X^{t,\phi'}, Y^{t',\phi}, Z^{t',\phi} \right) \\
& \quad \cdot \mathbb{1}_{\{(X^{t,\phi}, Y^{t',\phi}), (X^{t,\phi'}, Y^{t',\phi}) \in \mathcal{K}_\epsilon \times \mathcal{K}'_\epsilon, |Z^{t',\phi}| \leq \kappa, \|X^{t,\phi} - X^{t,\phi'}\|_T \leq \epsilon'\}} dr \\
& \quad + \mathbb{E} \int_0^T \Phi \left(X^{t,\phi}, X^{t,\phi'}, Y^{t',\phi}, Z^{t',\phi} \right) \mathbb{1}_{\{(X^{t,\phi}, Y^{t',\phi}, Z^{t',\phi}(r)) \notin \mathcal{K}_\epsilon \times \mathcal{K}'_\epsilon\}} dr \\
& \quad + \mathbb{E} \int_0^T \Phi \left(X^{t,\phi}, X^{t,\phi'}, Y^{t',\phi}, Z^{t',\phi} \right) \mathbb{1}_{\{(X^{t,\phi}, Y^{t',\phi}, Z^{t',\phi}(r)) \notin \mathcal{K}_\epsilon \times \mathcal{K}'_\epsilon\}} dr \\
& \quad + \mathbb{E} \int_0^T \Phi \left(X^{t,\phi}, X^{t,\phi'}, Y^{t',\phi}, Z^{t',\phi} \right) \mathbb{1}_{\{|Z^{t',\phi}(r)| > \kappa\}} dr \\
& \quad + \mathbb{E} \int_0^T \Phi \left(X^{t,\phi}, X^{t,\phi'}, Y^{t',\phi}, Z^{t',\phi} \right) \mathbb{1}_{\{\|X^{t,\phi} - X^{t,\phi'}\|_T > \epsilon'\}} dr
\end{aligned}$$

and therefore

$$\begin{aligned}
& \mathbb{E} \int_0^T \Phi \left(r, X^{t,\phi}, X^{t,\phi'}, Y^{t',\phi}, Z^{t',\phi}(r) \right) dr \\
& \leq T m_{h,f}(\epsilon', \mathcal{K}_\epsilon, \mathcal{U}_\epsilon, \kappa) + 2 \left\{ \mathbb{E} \left[\int_0^T \Phi \left(r, X^{t,\phi}, X^{t,\phi'}, Y^{t',\phi}, Z^{t',\phi}(r) \right) dr \right]^{p'} \right\}^{1/p'} \epsilon^{1-\frac{1}{p'}} \\
& \quad + C \mathbb{E} \left[\left(1 + \|X^{t,\phi}\|_T^{2p} + \|X^{t,\phi'}\|_T^{2p} + \|Y^{t',\phi}\|_T^2 \right) \int_0^T \mathbb{1}_{\{|Z^{t',\phi}(r)| > \kappa\}} dr \right] \\
& \quad + C \mathbb{E} \int_0^T |Z^{t',\phi}(r)|^2 \mathbb{1}_{\{|Z^{t',\phi}(r)| > \kappa\}} dr \\
& \quad + \left\{ \mathbb{E} \left[\int_0^T \Phi \left(r, X^{t,\phi'}, X^{t,\phi''}, Y^{t',\phi}, Z^{t',\phi}(r) \right) dr \right]^{p'} \right\}^{1/p'} \\
& \quad \cdot \left[\mathbb{P} \left(\|X^{t,\phi} - X^{t,\phi'}\|_T > \epsilon' \right) \right]^{1-\frac{1}{p'}} \\
& \leq T m_{h,f}(\epsilon', \mathcal{K}_\epsilon, \mathcal{U}, \kappa) \\
& \quad + C \left(1 + \|\phi\|_T^p + \|\phi'\|_T^p \right) \left[\epsilon^{1-\frac{1}{p'}} + \frac{\mathbb{E} \|X^{t,\phi} - X^{t,\phi'}\|_T^{p'-1}}{(\epsilon')^{p'-1}} + \frac{(\mathbb{E} \int_0^T |Z^{t',\phi}(r)|^2 dr)^{1/2}}{\kappa^2} \right] \\
& \quad + C \mathbb{E} \int_0^T |Z^{t',\phi}(r)|^2 \mathbb{1}_{\{|Z^{t',\phi}(r)| > \kappa\}} dr.
\end{aligned}$$

Eventually, by Theorem 4, we have

$$\begin{aligned}
& \sup_{t \in [0, T]} \mathbb{E} \int_0^T \Phi \left(r, X^{t, \phi}, X^{t, \phi'}, Y^{t', \phi}, Z^{t', \phi}(r) \right) dr \\
& \leq T m_{h, f}(\epsilon', \mathcal{K}_\epsilon, \mathcal{U}_\epsilon, \kappa) \\
& \quad + C \left(1 + \|\phi\|_T^p + \|\phi'\|_T^p \right) \left[\epsilon^{1 - \frac{1}{p'}} + \frac{\left(1 + \|\phi\|_T^{p-1} + \|\phi'\|_T^{p-1} \right) \mathbb{E} \|\phi - \phi'\|_T^{p'-1}}{(\epsilon')^{p'-1}} + \frac{1}{\kappa^2} \right] \\
& \quad + C \mathbb{E} \int_0^T |Z^{t', \phi}(r)|^2 \mathbb{1}_{\{|Z^{t', \phi}(r)| > \kappa\}} dr.
\end{aligned}$$

Passing now to the limit as $\phi' \rightarrow \phi$, $\epsilon' \rightarrow 0$, $(\epsilon, \kappa) \rightarrow (0, +\infty)$, we obtain the claim.

Concerning the continuity of $[0, T] \ni t \rightarrow u(t, \phi)$, this is an immediate consequence of the continuity of the stochastic process $Y^{t, \phi}$, together with the continuity of the mapping $t \mapsto Y^{t, \phi}$ from $[0, T]$ into $\mathcal{S}_0^{2,1}$. \blacksquare

In order to prove the generalized Feynman–Kac formula suitable for our framework we first consider the particular case when the generator f is independent of the past values of Y and Z , namely $(Y^{t, \phi}, Z^{t, \phi})$ is the solution of the standard BSDE with Lipschitz coefficients

$$Y^{t, \phi}(s) = h(X^{t, \phi}) + \int_s^T f(r, X^{t, \phi}, Y^{t, \phi}(r), Z^{t, \phi}(r)) dr - \int_s^T Z^{t, \phi}(r) dW(r), \quad s \in [t, T]. \quad (41)$$

Theorem 13 *Let us assume that assumptions $(A_1), (A_2), (A_6), (A_7)$ hold. Then there exists a continuous non-anticipative functional $u : [0, T] \times \mathbb{A} \rightarrow \mathbb{R}$ such that*

$$Y^{t, \phi}(s) = u(s, X^{t, \phi}), \quad \text{for all } s \in [t, T], \quad a.s.,$$

for every $(t, \phi) \in [0, T] \times \mathbb{A}$.

Proof. Again, for the sake of readability, we split the proof into several steps.

Step I.

Let $0 = t_0 < t_1 < \dots < t_n = T$ and suppose that

$$\begin{aligned}
b(t, \phi) &= b_1(t, \phi(t)) \mathbb{1}_{[0, t_1)}(t) + b_2(t, \phi(t_1), \phi(t) - \phi(t_1)) \mathbb{1}_{[t_1, t_2)}(t) + \dots \\
&\quad + b_n(t, \phi(t_1), \phi(t_2) - \phi(t_1), \dots, \phi(t) - \phi(t_{n-1})) \mathbb{1}_{[t_{n-1}, T]}(t), \\
\sigma(t, \phi) &= \sigma_1(t, \phi(t)) \mathbb{1}_{[0, t_1)}(t) + \sigma_2(t, \phi(t_1), \phi(t) - \phi(t_1)) \mathbb{1}_{[t_1, t_2)}(t) + \dots \\
&\quad + \sigma_n(t, \phi(t_1), \phi(t_2) - \phi(t_1), \dots, \phi(t) - \phi(t_{n-1})) \mathbb{1}_{[t_{n-1}, T]}(t)
\end{aligned}$$

and

$$\begin{aligned}
f(t, \phi, y, z) &= f_1(t, \phi(t), y, z) \mathbb{1}_{[0, t_1)}(t) + f_2(t, \phi(t_1), \phi(t) - \phi(t_1), y, z) \mathbb{1}_{[t_1, t_2)}(t) + \dots \\
&\quad + f_n(t, \phi(t_1), \phi(t_2) - \phi(t_1), \dots, \phi(t) - \phi(t_{n-1})) \mathbb{1}_{[t_{n-1}, T]}(t), \\
h(\phi) &= \varphi(\phi(t_1), \phi(t_2) - \phi(t_1), \dots, \phi(T) - \phi(t_{n-1})),
\end{aligned}$$

for every $\phi \in \Lambda$.

Let us first show that the terms $(X^{t,\phi}(t_1), \dots, X^{t,\phi}(r) - X^{t,\phi}(t_k)), 0 \leq k \leq n-1$ are related to the solution of a SDE equation of Itô type in $\mathbb{R}^{n \times d}$. Let

$$\begin{aligned} \tilde{b}(t, x_1, \dots, x_n) &:= \begin{pmatrix} b_1(t, x_1) \mathbb{1}_{[0, t_1)}(t) \\ b_2(t, x_1, x_2) \mathbb{1}_{[t_1, t_2)}(t) \\ \vdots \\ b_n(t, x_1, \dots, x_n) \mathbb{1}_{[t_{n-1}, t_n)}(t) \end{pmatrix}, \\ \tilde{\sigma}(t, x_1, \dots, x_n) &:= \begin{pmatrix} \sigma_1(t, x_1) \mathbb{1}_{[0, t_1)}(t) \\ \sigma_2(t, x_1, x_2) \mathbb{1}_{[t_1, t_2)}(t) \\ \vdots \\ \sigma_n(t, x_1, \dots, x_n) \mathbb{1}_{[t_{n-1}, t_n)}(t) \end{pmatrix}. \end{aligned}$$

Let then $\tilde{X}^{t, \mathbf{x}}$, with $t \in [0, T]$ and $\mathbf{x} = (x_1, \dots, x_n) \in \mathbb{R}^{d \times n}$ be the unique solution of the following stochastic differential equation:

$$\tilde{X}^{t, \mathbf{x}}(s) = \mathbf{x} + \int_t^s \tilde{b}(r, \tilde{X}^{t, \mathbf{x}}) dr + \int_t^s \tilde{\sigma}(r, \tilde{X}^{t, \mathbf{x}}) dW(r), \quad s \in [t, T].$$

We assert that, for $t \in [t_{k_0}, t_{k_0+1})$, $s \in [t_k, t_{k+1}]$, $s \geq t$, with $0 \leq k_0 \leq k \leq n-1$, we have

$$(X^{t,\phi}(t_1), \dots, X^{t,\phi}(s) - X^{t,\phi}(t_k)) = \left(\tilde{X}^{i, t, (\phi(t_1), \dots, \phi(t) - \phi(t_{k_0}), 0, \dots, 0)}(s) \right)_{i=\overline{1, k+1}}.$$

Let us stress that for $k = 0$ this reads $X^{t,\phi}(s) = \tilde{X}^{1, t, \phi(t)}(s)$; so that for $k > k_0 = 0$, it is interpreted as $(X^{t,\phi}(t_1), \dots, X^{t,\phi}(s) - X^{t,\phi}(t_k)) = (\tilde{X}^{i, t, (\phi(t_1), \dots, \phi(t) - \phi(t_{k_0}), 0, \dots, 0)}(s))_{i=\overline{1, k+1}}$.

We will prove this statement by induction on k . If $k = 0$, then $k_0 = 0$ and we obviously have

$$X^{t,\phi}(s) = \tilde{X}^{1, t, \phi(t)}(s).$$

Let us suppose that the statement holds true for $k-1$, for the sake of brevity, in what follows we will denote by $\mathbf{x} := (\phi(t_1), \dots, \phi(t) - \phi(t_{k_0}), 0, \dots, 0)$.

If $k_0 \leq k-1$ then, from the induction hypothesis, we have that

$$(X^{t,\phi}(t_1), \dots, X^{t,\phi}(r) - X^{t,\phi}(t_{k-1})) = \left(\tilde{X}^{i, t, (\phi(t_1), \dots, \phi(t) - \phi(t_{k_0}), 0, \dots, 0)}(r) \right)_{i=\overline{1, k}}$$

for every $r \in [t_{k-1}, t_k]$, so that, for $s \in [t_k, t_{k+1}]$ we have,

$$\tilde{X}^{j, t, \mathbf{x}}(s) = \tilde{X}^{j, t, \mathbf{x}}(t_k) = X^{t,\phi}(t_j) - X^{t,\phi}(t_{j-1}), \quad 1 \leq j \leq k,$$

with the convention $X^{t,\phi}(t_0) = 0$.

In the case $k_0 = k$, for $s \in [t, t_{k+1}]$ we also have:

$$\tilde{X}^{j, t, \mathbf{x}}(s) = \mathbf{x}^j = \phi(t_j) - \phi(t_{j-1}) = X^{t,\phi}(t_j) - X^{t,\phi}(t_{j-1}), \quad 1 \leq j \leq k,$$

again with the convention $\phi(t_0) = 0$.

Consequently, on $[t \vee t_k, t_{k+1}]$, it holds,

$$\begin{aligned}\tilde{X}^{k+1,t,\mathbf{x}}(s) &= \mathbf{x}^{k+1} + \int_{t \vee t_k}^s b_{k+1}(r, \tilde{X}^{1,t,\mathbf{x}}(r), \dots, \tilde{X}^{k+1,t,\mathbf{x}}(r)) dr \\ &\quad + \int_{t \vee t_k}^s \sigma_{k+1}(r, \tilde{X}^{1,t,\mathbf{x}}(r), \dots, \tilde{X}^{k+1,t,\mathbf{x}}(r)) dW(r) \\ &= \mathbf{x}^{k+1} + \int_{t \vee t_k}^s b_{k+1}(r, X^{t,\phi}(t_1), \dots, X^{t,\phi}(t_k) - X^{t,\phi}(t_{k-1}), \tilde{X}^{k+1,t,\mathbf{x}}(r)) dr \\ &\quad + \int_{t \vee t_k}^s \sigma_{k+1}(r, X^{t,\phi}(t_1), \dots, X^{t,\phi}(t_k) - X^{t,\phi}(t_{k-1}), \tilde{X}^{k+1,t,\mathbf{x}}(r)) dW(r).\end{aligned}$$

If $k_0 \leq k-1$ then $\mathbf{x}^{k+1} = 0$; if $k_0 = k$, then $\mathbf{x}^{k+1} = \phi(t) - \phi(t_{k_0}) = X^{t,\phi}(t) - X^{t,\phi}(t_k)$. By uniqueness, since $X^{t,\phi}$ satisfies

$$\begin{aligned}X^{t,\phi}(s) &= X^{t,\phi}(t \vee t_k) + \int_{t \vee t_k}^s b_{k+1}(r, X^{t,\phi}(t_1), \dots, X^{t,\phi}(r) - X^{t,\phi}(t_k)) dr \\ &\quad + \int_{t \vee t_k}^s \sigma_{k+1}(r, X^{t,\phi}(t_1), \dots, X^{t,\phi}(r) - X^{t,\phi}(t_k)) dW(r), \quad s \in [t \vee t_k, t_{k+1}].\end{aligned}$$

we obtain $\tilde{X}^{k+1,t,\mathbf{x}}(s) = X^{t,\phi}(s) - X^{t,\phi}(t_k)$, for all $s \in [t \vee t_k, t_{k+1}]$. We thus have proved that the statement holds true for k .

The next step is to derive a Feynman–Kac type formula linking $X^{t,\phi}$ and $Y^{t,\phi}$. For that, let us consider the following BSDE:

$$\tilde{Y}^{t,\mathbf{x}}(s) = \varphi(\tilde{X}^{t,\mathbf{x}}(T)) + \int_t^T \tilde{f}(r, \tilde{X}^{t,\mathbf{x}}(r), \tilde{Y}^{t,\mathbf{x}}(r), \tilde{Z}^{t,\mathbf{x}}(r)) dr + \int_t^s \tilde{Z}^{t,\mathbf{x}} dW(r), \quad s \in [t, T],$$

where \tilde{f} is defined by

$$\begin{aligned}\tilde{f}(t, x_1, \dots, x_n, y, z) \\ = f_1(t, x_1, y, z) \mathbb{1}_{[0, t_1)}(t) + f_2(t, x_1, x_2, y, z) \mathbb{1}_{[t_1, t_2)}(t) + \dots + f_n(t, x_1, x_2, \dots, x_n) \mathbb{1}_{[t_{n-1}, T]}(t).\end{aligned}$$

From [19], there exist some measurable functions \tilde{u}, \tilde{d} such that for all $(t, \mathbf{x}) \in [0, T] \times \mathbb{R}^{d \times n}$ it holds,

$$\begin{aligned}\tilde{Y}^{t,\mathbf{x}}(s) &= \tilde{u}(s, \tilde{X}^{t,\mathbf{x}}(s)), \quad \text{for all } s \in [t, T]; \\ \tilde{Z}^{t,\mathbf{x}}(s) &= \tilde{d}(s, \tilde{X}^{t,\mathbf{x}}(s)) \tilde{\sigma}(s, \tilde{X}^{t,\mathbf{x}}), \quad ds\text{-a.e. on } [t, T].\end{aligned}$$

On the other hand, let $t \in [t_{k_0}, t_{k_0+1})$, with $0 \leq k_0 \leq n-1$ and denote, for simplicity, $\mathbf{x} = (\phi(t_1), \dots, \phi(t) - \phi(t_{k_0}), 0, \dots, 0)$.

If $s \in [t_k, t_{k+1}]$, $s \geq t$, and therefore $k \geq k_0$, we have

$$(X^{t,\phi}(t_1), \dots, X^{t,\phi}(s) - X^{t,\phi}(t_k)) = \left(\tilde{X}^{i,t,\mathbf{x}}(s) \right)_{i=\overline{1, k+1}}$$

Thus, on $[t \vee t_k, t_{k+1}]$

$$\begin{aligned}\tilde{f}(s, \tilde{X}^{t,\mathbf{x}}(s), \tilde{Y}^{t,\mathbf{x}}(s), \tilde{Z}^{t,\mathbf{x}}(s)) &= f_{k+1}(s, X^{t,\phi}(t_1), \dots, X^{t,\phi}(s) - X^{t,\phi}(t_k), \tilde{Y}^{t,\mathbf{x}}(s), \tilde{Z}^{t,\mathbf{x}}(s)) \\ &= f(s, X^{t,\phi}, \tilde{Y}^{t,\mathbf{x}}(s), \tilde{Z}^{t,\mathbf{x}}(s)).\end{aligned}$$

Allowing k to vary, we obtain the equality

$$\tilde{f}(s, \tilde{X}^{t,\mathbf{x}}(s), \tilde{Y}^{t,\mathbf{x}}(s), \tilde{Z}^{t,\mathbf{x}}(s)) = f(s, X^{t,\phi}, \tilde{Y}^{t,\mathbf{x}}(s), \tilde{Z}^{t,\mathbf{x}}(s)), \quad \text{for all } s \in [t, T].$$

Also,

$$\varphi(\tilde{X}^{t,\mathbf{x}}(T)) = \varphi(X^{t,\phi}(t_1), \dots, X^{t,\phi}(T) - X^{t,\phi}(t_{n-1})) = h(X^{t,\phi}),$$

and by the uniqueness of the solution of the BSDE, we get that

$$(\tilde{Y}^{t,\mathbf{x}}, \tilde{Z}^{t,\mathbf{x}}) = (Y^{t,\phi}, Z^{t,\phi})$$

and, consequently,

$$\begin{aligned}Y^{t,\phi}(s) &= \tilde{u}(s, X^{t,\phi}(t_1), \dots, X^{t,\phi}(s) - X^{t,\phi}(t_k), 0, \dots, 0), \quad \text{for all } s \in [t \vee t_k, t_{k+1}], \\ Z^{t,\phi}(s) &= \tilde{d}_{k+1}(s, X^{t,\phi}(t_1), \dots, X^{t,\phi}(s) - X^{t,\phi}(t_k), 0, \dots, 0) \\ &\quad \cdot \sigma_{k+1}(s, X^{t,\phi}(t_1), \dots, X^{t,\phi}(s) - X^{t,\phi}(t_k)), \quad ds\text{-a.e. on } [t \vee t_k, t_{k+1}].\end{aligned}$$

By setting, for $0 \leq k \leq n-1$, $t \in [t_k, t_{k+1})$ and $\phi \in \Lambda$,

$$\begin{aligned}u(t, \phi) &:= \tilde{u}(t, \phi(t_1), \dots, \phi(t) - \phi(t_k), 0, \dots, 0); \\ d(t, \phi) &:= \tilde{d}_{k+1}(t, \phi(t_1), \dots, \phi(t) - \phi(t_k), 0, \dots, 0),\end{aligned}$$

we get that, for every $t \in [0, T]$ and $\phi \in \Lambda$

$$\begin{aligned}Y^{t,\phi}(s) &= u(s, X^{t,\phi}), \quad \text{for all } s \in [t, T]; \\ Z^{t,\phi}(s) &= d(s, X^{t,\phi})\sigma(t, X^{t,\phi}), \quad ds\text{-a.e. on } [t, T].\end{aligned}$$

Step II.

Let us notice that the same conclusion holds for b, σ and f of the form

$$\begin{aligned}b(t, \phi) &= b_1(t, \phi(t))\mathbf{1}_{[0, t_1)}(t) + b_2(t, \phi(t_1), \phi(t))\mathbf{1}_{[t_1, t_2)}(t) + \dots \\ &\quad + b_n(t, \phi(t_1), \dots, \phi(t_{n-1}), \phi(t))\mathbf{1}_{[t_{n-1}, T]}(t), \\ \sigma(t, \phi) &= \sigma_1(t, \phi(t))\mathbf{1}_{[0, t_1)}(t) + \sigma_2(t, \phi(t_1), \phi(t))\mathbf{1}_{[t_1, t_2)}(t) + \dots \\ &\quad + \sigma_n(t, \phi(t_1), \dots, \phi(t_{n-1}), \phi(t))\mathbf{1}_{[t_{n-1}, T]}(t)\end{aligned}$$

and

$$\begin{aligned}f(t, \phi, y, z) &= f_1(t, \phi(t), y, z)\mathbf{1}_{[0, t_1)}(t) + f_2(t, \phi(t_1), \phi(t), y, z)\mathbf{1}_{[t_1, t_2)}(t) + \dots \\ &\quad + f_n(t, \phi(t_1), \dots, \phi(t_{n-1}), \phi(t), y, z)\mathbf{1}_{[t_{n-1}, T]}(t), \\ h(\phi) &= \varphi(\phi(t_1), \dots, \phi(t_{n-1}), \phi(T)),\end{aligned}$$

for every $\phi \in \Lambda$.

Step III.

For $0 = t_0 \leq t_1 < \dots < t_k \leq T$ and $x_1, \dots, x_k \in \mathbb{R}^d$, let $\Phi_{t_1, \dots, t_k}^{x_1, \dots, x_k} : [0, T] \rightarrow \mathbb{R}^d$ be such that

$$\Phi_{t_1, \dots, t_k}^{x_1, \dots, x_k}(t_i) = x_i, \quad i = \overline{1, k}; \quad \Phi_{t_1, \dots, t_k}^{x_1, \dots, x_k}(T) = x_k, \quad \Phi_{t_1, \dots, t_k}^{x_1, \dots, x_k}(0) = x_1$$

and is prolonged to $[0, T]$ by linear interpolation.

Let us consider partitions of $[0, T]$, $0 = t_0^n < t_1^n < \dots < t_n^n = T$, $t_k^n := \frac{kT}{n}$. For $k \in \{1, \dots, n\}$, $t \in [0, T]$ and $x_1, \dots, x_k \in \mathbb{R}^d$, we define

$$b_k^n(t, x_1, \dots, x_k) := b(t, \Phi_{t_1^n, \dots, t_k^n}^{x_1, \dots, x_k}).$$

Notice that $b_k^n(t, \cdot)$ are continuous functions.

Finally, for $t \in [0, T]$ and $x_1, \dots, x_n \in \mathbb{R}^d$ we set

$$\bar{b}_n(t, x_1, \dots, x_n) := b_1^n(t, x_1) \mathbb{1}_{[0, t_1^n)} + \dots + b_n^n(t, x_1, \dots, x_n) \mathbb{1}_{[t_{n-1}^n, t_n^n]}$$

and, for $(t, \phi) \in [0, T] \times \Lambda$,

$$b_n(t, \phi) := \bar{b}_n(t, \phi(t \wedge t_1^n), \dots, \phi(t \wedge t_n^n)).$$

If $t \in [t_{k-1}^n, t_k^n)$ with $k \in \{1, \dots, n\}$, or if $t = T$ and $k = n$, then

$$b_n(t, \phi) = b_k^n(t, \phi(t_1^n), \dots, \phi(t_{k-1}^n), \phi(t)),$$

so b_n has the form described in **Step II**. Also in this case it holds

$$\begin{aligned} |b_n(t, \phi) - b(t, \phi)| &= |b(t, \Phi_{t_1^n, \dots, t_{k-1}^n, t_k^n}^{\phi(t_1^n), \dots, \phi(t_{k-1}^n), \phi(t)}) - b(t, \phi)| \\ &\leq \sup_{\|\psi - \phi\|_T \leq \omega(\phi, T/n)} |b(t, \psi) - b(t, \phi)| \leq \ell \omega(\phi, T/n), \end{aligned}$$

where $\omega(\phi, \epsilon) := \sup_{|s-r| \leq \epsilon} |\phi(s) - \phi(r)|$.

In a similar way, one introduces σ_n , f_n and h_n and we have, for every $(t, \phi, y, z) \in [0, T] \times \Lambda \times \mathbb{R} \times \mathbb{R}^{d'}$,

$$\begin{aligned} |\sigma_n(t, \phi) - \sigma(t, \phi)| &\leq \ell \omega(\phi, T/n); \\ |f_n(t, \phi, y, z) - f(t, \phi, y, z)| &\leq \sup_{\|\psi - \phi\|_T \leq \omega(\phi, T/n)} |f(t, \psi, y, z) - f(t, \phi, y, z)|; \\ |h_n(\phi) - h(\phi)| &\leq \sup_{\|\psi - \phi\|_T \leq \omega(\phi, T/n)} |h(\psi) - h(\phi)|. \end{aligned}$$

We also have that b_n , σ_n , f_n and h_n satisfy assumptions $(A_1), (A_2), (A_6), (A_7)$ with the same constants. Corresponding to these coefficients, we define the solution of the associated FB-SDE,

$$(X^{n,t,\phi}, Y^{n,t,\phi}, Z^{n,t,\phi}).$$

By the Feynman–Kac formula, already proven in this case, we have the existence of some non-anticipative functions u_n and d_n , such that, for every $(t, \phi) \in [0, T] \times \Lambda$, we have:

$$\begin{aligned} Y^{n,t,\phi}(s) &= u_n(s, X^{n,t,\phi}), \quad \text{for all } s \in [t, T]; \\ Z^{n,t,\phi}(s) &= d_n(s, X^{n,t,\phi}) \sigma(t, X^{n,t,\phi}), \quad ds\text{-a.e. on } [t, T]. \end{aligned}$$

In order to pass to the limit in the first formula above, we need to show that $X^{n,t,\phi} \rightarrow X^{t,\phi}$ in probability in Λ and that u_n converges to u on compact subsets of Λ .

Let $t \in [0, T]$ and $\phi, \phi' \in \Lambda$. By Itô's formula we have that,

$$\begin{aligned} & |X^{n,t,\phi'}(s) - X^{t,\phi}(s)|^2 = |\phi'(t) - \phi(t)|^2 \\ & + 2 \int_t^s \langle b_n(r, X^{n,t,\phi'}) - b(r, X^{t,\phi}), X^{n,t,\phi'}(r) - X^{t,\phi}(r) \rangle dr \\ & + \int_t^s |\sigma_n(r, X^{n,t,\phi'}) - \sigma(r, X^{t,\phi})|^2 dr \\ & + 2 \int_t^s \langle \sigma_n(r, X^{n,t,\phi'}) - \sigma(r, X^{t,\phi}), (X^{n,t,\phi'}(r) - X^{t,\phi}(r)) dW(r) \rangle. \end{aligned}$$

By standard calculations using Gronwall's lemma, Burkholder-Davis-Gundy inequalities and Theorem 4, we get, for some arbitrary $p > 1$,

$$\begin{aligned} & \mathbb{E} \sup_{s \in [0, T]} |X^{n,t,\phi'}(s) - X^{t,\phi}(s)|^{2p} \leq C \left(\|\phi' - \phi\|_T^{2p} + \mathbb{E} \omega(X^{t,\phi}, T/n)^{2p} \right) \\ & \leq C \left(\|\phi' - \phi\|_T^{2p} + \frac{1 + \|\phi\|_T^{2p}}{n^{p-1}} + \omega(\phi, T/n)^{2p} \right), \end{aligned} \quad (42)$$

where the positive constant C is depending only on p and the parameters of our FBSDE, also as above the constant C is allowed to change value from line to line during this proof.

Applying now Itô's formula to $e^{\beta s} |Y^{n,t,\phi'}(s) - Y^{t,\phi}(s)|^2$, we obtain for $\beta > 0$,

$$\begin{aligned} & e^{\beta s} |\Delta_Y^{n,t,\phi,\phi'}(s)|^2 + \beta \int_s^T e^{\beta r} |\Delta_Z^{n,t,\phi,\phi'}(r)|^2 dr + \int_s^T e^{\beta r} |\Delta_Z^{n,t,\phi,\phi'}(r)|^2 dr \\ & = e^{\beta T} |\Delta_h^{n,t,\phi,\phi'}|^2 + 2 \int_s^T \langle \Delta_f^{n,t,\phi,\phi'}(r), e^{\beta r} \Delta_Y^{n,t,\phi,\phi'}(r) \rangle dr \\ & + 2 \int_s^T \langle f_n(s, X^{n,t,\phi'}, Y^{n,t,\phi'}(s), Z^{n,t,\phi'}(s)) \\ & \quad - f_n(s, X^{n,t,\phi'}, Y^{t,\phi}(s), Z^{t,\phi}(s)), e^{\beta r} \Delta_Y^{n,t,\phi,\phi'}(r) \rangle dr \\ & - 2 \int_s^T \langle e^{\beta r} \Delta_Y^{n,t,\phi,\phi'}(r), \Delta_Z^{n,t,\phi,\phi'}(r) dW(r) \rangle, \end{aligned}$$

where, for the sake of simplicity, we have denoted

$$\begin{aligned} \Delta_Y^{n,t,\phi,\phi'}(s) &:= Y^{n,t,\phi'}(s) - Y^{t,\phi}(s); \\ \Delta_Z^{n,t,\phi,\phi'}(s) &:= Z^{n,t,\phi'}(s) - Z^{t,\phi}(s); \\ \Delta_h^{n,t,\phi,\phi'} &:= h_n(X^{n,t,\phi'}) - h(X^{t,\phi}); \\ \Delta_f^{n,t,\phi,\phi'}(s) &:= f_n(s, X^{n,t,\phi'}, Y^{t,\phi}(s), Z^{t,\phi}(s)) - f(s, X^{t,\phi}, Y^{t,\phi}(s), Z^{t,\phi}(s)). \end{aligned}$$

Again, for β sufficiently large, exploiting Burkholder-Davis-Gundy inequalities and the Lipschitz property in $(y, z) \in \mathbb{R} \times \mathbb{R}^d$ of f_n , we infer that

$$\mathbb{E} \sup_{s \in [t, T]} |\Delta_Y^{n, t, \phi, \phi'}(s)|^2 + \mathbb{E} \int_t^T |\Delta_Z^{n, t, \phi, \phi'}(r)|^2 dr \leq C \mathbb{E} \left[|\Delta_h^{n, t, \phi, \phi'}|^2 + \int_t^T |\Delta_f^{n, t, \phi, \phi'}(r)|^2 dr \right]. \quad (43)$$

We now have

$$\begin{aligned} |\Delta_h^{n, t, \phi, \phi'}| &\leq |h_n(X^{n, t, \phi'}) - h(X^{n, t, \phi'})| + |h(X^{n, t, \phi'}) - h(X^{t, \phi})| \\ &\leq \sup_{\|\psi - X^{n, t, \phi'}\|_T \leq \omega(X^{n, t, \phi'}, T/n)} |h(\psi) - h(X^{n, t, \phi'})| + |h(X^{n, t, \phi'}) - h(X^{t, \phi})| \\ &\leq \sup_{\|\psi - X^{t, \phi}\|_T \leq \omega(X^{t, \phi}, T/n) + 3\|X^{n, t, \phi'} - X^{t, \phi}\|_T} |h(\psi) - h(X^{t, \phi})| + 2|h(X^{n, t, \phi'}) - h(X^{t, \phi})| \\ &\leq 3 \sup_{\|\psi - X^{t, \phi}\|_T \leq \omega(X^{t, \phi}, T/n) + 3\|X^{n, t, \phi'} - X^{t, \phi}\|_T} |h(\psi) - h(X^{t, \phi})|, \end{aligned} \quad (44)$$

since

$$\omega(X^{n, t, \phi'}, T/n) \leq \omega(X^{t, \phi}, T/n) + 2\|X^{n, t, \phi'} - X^{t, \phi}\|_T.$$

Similarly, we obtain,

$$|\Delta_f^{n, t, \phi, \phi'}(s)| \leq 3 \sup_{\|\psi - X^{t, \phi}\|_T \leq \omega(X^{t, \phi}, T/n) + 3\|X^{n, t, \phi'} - X^{t, \phi}\|_T} |f(s, \psi, Y^{t, \phi}(s), Z^{t, \phi}(s)) - f(s, X^{t, \phi}, Y^{t, \phi}(s), Z^{t, \phi}(s))|. \quad (45)$$

Let now $(t, \phi) \in [0, T] \times \Lambda$ and let (ϕ_n) be a sequence converging to ϕ in Λ . It is clear from relation (42) that (X^{n, t, ϕ_n}) converges in $L^{2p}(\Omega; \Lambda)$ to $X^{t, \phi}$, therefore there exists a subsequence converging a.s. in Λ to $X^{t, \phi}$. Without restricting the generality, we will still denote (X^{n, t, ϕ_n}) this subsequence. Since $\omega(X^{t, \phi}, T/n) + 3\|X^{n, t, \phi'} - X^{t, \phi}\|_T$ converges to 0 a.s., it is clear by relations (44) and (42) that $\Delta_h^{n, t, \phi, \phi'}$ and $\Delta_f^{n, t, \phi, \phi'}(s)$ converge to 0, a.s., respectively $ds\mathbb{P}$ -a.e.

Then, by the dominated convergence theorem, we obtain

$$\mathbb{E} \left[|\Delta_h^{n, t, \phi, \phi_n}|^2 + \int_t^T |\Delta_f^{n, t, \phi, \phi_n}(r)|^2 dr \right] \rightarrow 0,$$

which, combined with estimate (43), gives that

$$\mathbb{E} \sup_{s \in [t, T]} |Y^{n, t, \phi_n}(s) - Y^{t, \phi}(s)|^2 \rightarrow 0,$$

which implies, assuming $\phi_n \equiv \phi$

$$\mathbb{E} \sup_{s \in [t, T]} |Y^{n, t, \phi}(s) - Y^{t, \phi}(s)|^2 \rightarrow 0,$$

and, letting $s = t$, we obtain,

$$u_n(t, \phi_n) \rightarrow u(t, \phi).$$

Therefore we can pass to the limit in the relation

$$Y^{n, t, \phi}(s) = u_n(s, X^{n, t, \phi}), \quad \text{for all } s \in [t, T], \text{ a.s.},$$

replacing in the right term ϕ_n by $X^{n,t,\phi}$ and ϕ by $X^{n,t,\phi}$, and find that

$$Y^{t,\phi}(s) = u(s, X^{t,\phi}), \quad \text{for all } s \in [t, T], \text{ a.s.}$$

■

The following result represents the Feynman–Kac formula adapted to our framework.

Theorem 14 *Let us assume that assumptions $(A_1), (A_2), (A_6), (A_7)$ hold and condition (12) is verified. Then there exists a continuous non-anticipative functional $u : [0, T] \times \mathbb{A} \rightarrow \mathbb{R}$ such that*

$$Y^{t,\phi}(s) = u(s, X^{t,\phi}), \quad \text{for all } s \in [0, T], \quad \text{a.s.,}$$

for any $(t, \phi) \in [0, T] \times \mathbb{A}$.

Proof. The continuity of u was already asserted in Theorem 12, so that the proof follows with the same arguments.

Let now consider BSDE with delayed generator (52)

$$Y^{t,\phi}(s) = h(X^{t,\phi}) + \int_s^T f(r, X^{t,\phi}, Y_r^{t,\phi}, Z_r^{t,\phi}, Y_r^{t,\phi}) dr - \int_s^T Z_r^{t,\phi} dW(r), \quad (46)$$

and the corresponding iterative equations

$$\begin{aligned} Y^{n+1,t,\phi}(s) = & h(X^{t,\phi}) + \int_s^T f(r, X^{t,\phi}, Y_r^{n+1,t,\phi}, Z_r^{n+1,t,\phi}, Y_r^{n,t,\phi}) dr \\ & - \int_s^T Z_r^{n+1,t,\phi} dW(r), \quad s \in [t, T], \end{aligned} \quad (47)$$

with $Y^{0,t,\phi} \equiv 0$ and $Z^{0,t,\phi} \equiv 0$.

Let us suppose that there exists $u_n : [0, T] \times \mathbb{A} \rightarrow \mathbb{R}$ a \mathbb{F} -progressively measurable functional such that u_n is continuous and

$$Y^{n,t,\phi}(s) = u_n(s, X^{t,\phi}),$$

for every $t, s \in [0, T]$ and $\phi \in \mathbb{A}$.

Let us now consider the term

$$Y_r^{n,t,\phi} = (Y^{n,t,\phi}(r + \theta))_{\theta \in [-\delta, 0]},$$

in particular we have that, if $r + \theta \geq 0$, then $Y^{n,t,\phi}(r + \theta) = u_n(r + \theta, X^{t,\phi})$ and if $r + \theta < 0$, then $Y^{n,t,\phi}(r + \theta) = Y^{n,t,\phi}(0) = u_n(0, X^{t,\phi})$. By defining then

$$\tilde{u}_n(t, \phi) := (u_n(\mathbb{1}_{[0,T]}(t + \theta), \phi))_{\theta \in [-\delta, 0]},$$

we have

$$Y_r^{n,t,\phi} = \tilde{u}_n(r, X^{t,\phi}).$$

Then we can apply Theorem 13 in order to infer that

$$Y^{n+1,t,\phi}(s) = u_{n+1}(s, X^{t,\phi}),$$

for a continuous non-anticipative functional $u_{n+1} : [0, T] \times \mathbb{A} \rightarrow \mathbb{R}$.

Notice that $(Y^{n,t,\phi}, Z^{n,t,\phi})$ is the Picard iteration sequence used for constructing the solution $(Y^{t,\phi}, Z^{t,\phi})$, recall that

$$(Y^{n+1,\cdot,\phi}, Z^{n+1,\cdot,\phi}) = \Gamma(Y^{n,\cdot,\phi}, Z^{n,\cdot,\phi}),$$

being Γ the contraction defined in the proof of Theorem 7, we have that

$$\lim_{n \rightarrow \infty} \mathbb{E} \left(\sup_{s \in [0, T]} |Y^{n,t,\phi}(s) - Y^{t,\phi}(s)|^2 \right) = 0.$$

Of course $u_n(t, \phi)$ converges to

$$u(t, \phi) := \mathbb{E} Y^{t,\phi}(t),$$

for every $t \in [0, T]$ and $\phi \in \mathbb{A}$. This implies that the nonlinear Feynman–Kac formula

$$Y^{t,\phi}(s) = u(s, X^{t,\phi})$$

holds. ■

We are now able to prove Theorem 11, which shows the existence of a viscosity solution to the PDKE (1).

Proof of Theorem 11.

We will prove that function u defined by (33) is a viscosity solution to (1). In particular we will only show that u is a viscosity subsolution, the supersolution case is similar.

Suppose, by contrary, that u is not a viscosity subsolution. Then, for any $L_0 \geq 0$, there exists $L \geq L_0$ such that u is not a viscosity L -subsolution in the sense of Definition 2. Therefore, there exists $(t, \phi) \in [0, T] \times \mathbb{A}$ and there exists $\varphi \in \underline{\mathcal{A}}^L u(t, \phi)$ such that

$$\partial_t \varphi(t, \phi) + \mathcal{L} \varphi(t, \phi) + f(t, \phi, \varphi(t, \phi), \partial_x \varphi(t, \phi) \sigma(t, \phi), (\varphi(\cdot, \phi))_t) \leq -c < 0$$

for some $c > 0$.

Using the definition of $\underline{\mathcal{A}}^L u(t, \phi)$ we see that there exists $\tau_0 \in \mathcal{T}_+^t$ such that

$$\varphi(t, \phi) - u(t, \phi) = \min_{\tau \in \mathcal{T}^t} \underline{\mathcal{E}}_t^L \left[(\varphi - u)(\tau \wedge \tau_0, X^{t,\phi}) \right],$$

and by the continuity of the coefficients, we deduce that there exists $\tilde{\tau} \in \mathcal{T}_+^t$ such that

$$\begin{aligned} & \partial_t \varphi(s, X^{t,\phi}) + \mathcal{L} \varphi(s, X^{t,\phi}) + \\ & + f(s, X^{t,\phi}, \varphi(s, X^{t,\phi}), \partial_x \varphi(s, X^{t,\phi}) \sigma(s, X^{t,\phi}), (\varphi(\cdot, X^{t,\phi}))_s) \leq -c/2 < 0, \end{aligned}$$

for any $s \in [t, \tilde{\tau}]$. Let us now take

$$\begin{aligned} \tilde{\tau} = T \wedge \tau_0 \wedge \inf \{ s > t : & \partial_t \varphi(s, X^{t,\phi}) + \mathcal{L} \varphi(s, X^{t,\phi}) \\ & + f(s, X^{t,\phi}, \varphi(s, X^{t,\phi}), \partial_x \varphi(s, X^{t,\phi}) \sigma(s, X^{t,\phi}), (\varphi(\cdot, X^{t,\phi}))_s) > c/2 \}. \end{aligned}$$

Let us denote

$$\begin{aligned}(Y^1(s), Z^1(s)) &:= (\varphi(s, X^{t,\phi}), \partial_x \varphi(s, X^{t,\phi}) \sigma(s, X^{t,\phi})), \quad \text{for any } s \in [t, T], \\ (Y^2(s), Z^2(s)) &:= (Y^{t,\phi}(s), Z^{t,\phi}(s)), \quad \text{for any } s \in [0, T]\end{aligned}$$

and

$$\Delta Y(s) := Y^1(s) - Y^2(s), \quad \Delta Z(s) := Z^1(s) - Z^2(s), \quad \text{for any } s \in [t, T].$$

Using Itô's formula we deduce, for any $s \in [t, T]$,

$$\varphi(s, X^{t,\phi}) = \varphi(t, \phi) + \int_t^s (\partial_t \varphi(r, X^{t,\phi}) + \mathcal{L}\varphi(r, X^{t,\phi})) dr + \int_t^s \langle \partial_x \varphi(r, X^{t,\phi}), \sigma(r, X^{t,\phi}) dW(r) \rangle$$

so that we obtain

$$\begin{aligned}\Delta Y(\tilde{\tau}) - \Delta Y(t) &= \int_t^{\tilde{\tau}} (\partial_t \varphi(r, X^{t,\phi}) + \mathcal{L}\varphi(r, X^{t,\phi}) + f(r, X^{t,\phi}, Y^2(r), Z^2(r), Y_r^2)) dr + \int_t^{\tilde{\tau}} \Delta Z(r) dW(r).\end{aligned}$$

Since for any $r \in [t, \tilde{\tau}]$ we have that

$$\begin{aligned}\partial_t \varphi(r, X^{t,\phi}) + \mathcal{L}\varphi(r, X^{t,\phi}) + \partial_t \varphi(r, X^{t,\phi}) + \mathcal{L}\varphi(r, X^{t,\phi}) + f(r, X^{t,\phi}, Y^2(r), Z^2(r), Y_r^2) \\ \leq -\frac{c}{2} + f(r, X^{t,\phi}, Y^2(r), Z^2(r), Y_r^2) \\ - f(r, X^{t,\phi}, \varphi(r, X^{t,\phi}), \partial_x \varphi(r, X^{t,\phi}) \sigma(r, X^{t,\phi}), (\varphi(\cdot, X^{t,\phi}))_r),\end{aligned}$$

we deduce, using Feynman–Kac formula, namely $Y^2(r) = u(r, X^{t,\phi})$, that,

$$\begin{aligned}\Delta Y(\tilde{\tau}) - \Delta Y(t) &\leq -\frac{c}{2}(\tilde{\tau} - t) + \int_t^{\tilde{\tau}} \Delta Z(r) dW(r) \\ &\quad + \int_t^{\tilde{\tau}} (f(r, X^{t,\phi}, \varphi(r, X^{t,\phi}), Z^2(r), (\varphi(\cdot, X^{t,\phi}))_r) \\ &\quad - f(r, X^{t,\phi}, \varphi(r, X^{t,\phi}), Z^1(r), (\varphi(\cdot, X^{t,\phi}))_r)) dr.\end{aligned}$$

We thus have that there exists $\lambda \in \mathcal{U}_T^L$ such that

$$\begin{aligned}f(r, X^{t,\phi}, \varphi(r, X^{t,\phi}), Z^1(r), (\varphi(\cdot, X^{t,\phi}))_r) - f(r, X^{t,\phi}, \varphi(r, X^{t,\phi}), Z^2(r), (\varphi(\cdot, X^{t,\phi}))_r) \\ = \langle \lambda(r), \Delta Z(r) \rangle,\end{aligned}$$

and therefore

$$\Delta Y(\tilde{\tau}) - \Delta Y(t) \leq -\frac{c}{2}(\tilde{\tau} - t) + \int_t^{\tilde{\tau}} \Delta Z(r) (dW(r) - \lambda(r) dr).$$

Noticing now that $W(s) - \int_t^s \lambda(r) dr$ is $\mathbb{P}^{t,\lambda}$ -martingale, we obtain that

$$\begin{aligned}
\Delta Y(t) &= \mathbb{E}^{\mathbb{P}^{t,\lambda}}(\Delta Y(t)) \\
&\geq \mathbb{E}^{\mathbb{P}^{t,\lambda}}(\Delta Y(\tilde{\tau})) + \frac{c}{2} \mathbb{E}^{\mathbb{P}^{t,\lambda}}(\tilde{\tau} - t) + \mathbb{E}^{\mathbb{P}^{t,\lambda}} \int_t^{\tilde{\tau}} \Delta Z(r) (dW(r) - \lambda(r) dr) \\
&> \mathbb{E}^{\mathbb{P}^{t,\lambda}}(\Delta Y(\tilde{\tau})) = \mathbb{E}^{\mathbb{P}^{t,\lambda}}(\varphi(\tilde{\tau}, X^{t,\phi}) - Y^{t,\phi}(\tilde{\tau})) = \mathbb{E}^{\mathbb{P}^{t,\lambda}}[(\varphi - u)(\tilde{\tau}, X^{t,\phi})] \\
&\geq \underline{\mathcal{E}}_t^L[(\varphi - u)(\tilde{\tau}, X^{t,\phi})] = \underline{\mathcal{E}}_t^L[(\varphi - u)(\tau_0 \wedge \tilde{\tau}, X^{t,\phi})] \\
&\geq \min_{\tau \in \mathcal{T}^t} \underline{\mathcal{E}}_t^L[(\varphi - u)(\tau_0 \wedge \tilde{\tau}, X^{t,\phi})] = \varphi(t, X^{t,\phi}) - Y^{t,\phi}(t),
\end{aligned}$$

which is a contradiction and the proof is thus complete. ■

5 Financial applications

In what follows we shall apply theoretical results developed in previous sections to analyze some particular models of great interest in modern finance.

Financial literature that shows how delay naturally arise when dealing with asset price evolution or in general with certain financial instruments, is nowadays wide and developed, see, for instance, [1, 5, 23, 24] and reference therein. On the other hand, not much is done when the delay enters the backward component. We aim here to give some financial applications where also the backward equation exhibits a delayed behaviour. We remark that since the goal of the present work is purely theoretical, the examples provided will not be stated in complete generality. In fact in this section we will show how the study of BSDEs with delayed generator, and the associated path-dependent Kolmogorov equation, may lead to the study of a completely new class of financial problems that have not been studied before. Nevertheless, we intend to address the study of these problems in a complete generality in a future work.

BSDEs with delay have been introduced in [12] as a pure mathematical tool with no financial application of interest. Later on, some works have appeared showing that the delay in the backward component arise naturally in several applications, see, e.g. [9, 10].

In what follows we will provide two extensions of forward-backward models that have been proposed in past literature where the backward components can exhibit a short-time delay.

5.1 The large investor problem

Following the model studied in [7], see also, e.g. [20], we will consider in the present example a non standard investor acting on a financial market. We assume this investor, usually referred to in literature as *the large investor*, has superior information about the stock prices and/or he is willing to invest a large amount of money in the stock. This fact implies that the large investor may influence the behaviour of the stock price with his actions. It is further natural to assume that there is a short time delay between the investor's actions and the reaction of the market to the large investor's actions. In particular we assume that the

drift coefficient of the underlying S at time t depends on how the large investor acts on the market in the interval $(t - \delta, t)$.

Let us then consider a risky asset S and a riskless bond B evolving according to

$$\begin{cases} \frac{dB(t)}{B(t)} = r(t, X(t), \pi(t), X_t) dt, & B(0) = 1, \\ \frac{dS(t)}{S(t)} = \mu(t, X(t), \pi(t), X_t) dt + \sigma(t, X(t), X_t) dW(t), & S(0) = s_0 > 0. \end{cases} \quad (48)$$

Here we have denoted by X the portfolio of the large investor. Also we used the notations introduced by (6) and (7). We suppose that the coefficients r, μ and σ satisfy some suitable regularity assumptions.

We have that the portfolio X , composed at any time $t \in [0, T]$ by $\pi(t)$, the amount of shares (held by the large investor) of the risky asset S and by $X(t) - \pi(t)$, the amount shares of the riskless bond B , evolves according to

$$\begin{aligned} dX(t) &= \frac{\pi(t)}{S(t)} dS(t) + \frac{X(t) - \pi(t)}{B(t)} dB(t) \\ &= \pi(t) \cdot [\mu(t, X(t), \pi(t), X_t) dt + \sigma(t, X(t), X_t) dW(t)] \\ &\quad + [X(t) - \pi(t)] \cdot r(t, X(t), \pi(t), X_t) dt, \end{aligned}$$

with the final condition $X(T) = h(S)$.

Hence, for $t \in [0, T]$,

$$X(t) = h(S) + \int_t^T F(s, X(s), \pi(s), X_s, \pi_s) ds - \int_t^T \pi(s) \sigma(s, X(s), X_s) dW(s), \quad (49)$$

where we have denoted for short

$$\begin{aligned} F(s, X(s), \pi(s), X_s, \pi_s) &:= -[X(s) - \pi(s)] \cdot r(s, X(s), \pi(s), X_s) \\ &\quad - \pi(s) \cdot \mu(s, X(s), \pi(s), X_s). \end{aligned} \quad (50)$$

Since the forward equations from (48) can be solved explicitly by

$$\begin{aligned} S(t) &= s_0 \exp \left[\int_0^t \left(\mu(s, X(s), \pi(s), X_s) - \frac{1}{2} \sigma^2(s, X(s), X_s) \right) ds \right. \\ &\quad \left. + \int_0^t \sigma(s, X(s), X_s) dW(s) \right], \end{aligned}$$

we deduce that S is a functional of X, π and W , i.e. there exists \tilde{h} such that the final condition becomes $X(T) = \tilde{h}(W, X, \pi)$.

A first remark is that we can impose some suitable assumptions on the functions r, μ and σ such that the function $\bar{F} : [0, T] \times \mathbb{R} \times \mathbb{R} \times L^2([-\delta, 0]; \mathbb{R}) \rightarrow \mathbb{R}$, defined by

$$\bar{F}(s, y, z, \hat{y}) := -(y - z\sigma^{-1}(s, y, \hat{y})) \cdot r(s, y, z\sigma^{-1}(s, y, \hat{y}), \hat{y}) - z \cdot \mu(s, y, z\sigma^{-1}(s, y, \hat{y}), \hat{y}),$$

satisfies assumptions (A₃)-(A₅).

The second remark concerns the fact that Theorem 7 is still true, with a slight adjustment of the proof, if we consider in the backward equation (8) the final condition

$$\bar{h}(W, X, Z) := \tilde{h}(W, X, \sigma^{-1}(\cdot, X, X) Z),$$

instead of a functional of W only, with \bar{h} satisfying a Lipschitz condition:

$$|\bar{h}(x, y, z) - \bar{h}(x, y', z')|^2 \leq L \left[\int_0^T |y(s) - y'(s)|^2 ds + \int_0^T |z(s) - z'(s)|^2 ds \right].$$

Therefore we can rewrite (49) as

$$X(t) = \bar{h}(W, X, Z) + \int_t^T \bar{F}(s, X(s), Z(s), X_s) ds - \int_t^T Z(s) dW(s), \quad t \in [0, T] \quad (51)$$

and we deduce from Theorem 7 that, under proper assumptions on the coefficients, there exists a unique solution (X, Z) to equation (51).

Hence equation (49) admits a unique solution (X, π) , where

$$\pi(s) := Z(s) \sigma^{-1}(s, X(s), X_s).$$

In order to obtain the connection with the PDE we consider first the problem

$$\left\{ \begin{array}{l} \bar{W}^{t,\phi}(s) = \phi(t) + \int_t^s dW(r), \quad s \in [t, T], \\ \bar{W}^{t,\phi}(s) = \phi(s), \quad s \in [0, t), \\ X^{t,\phi}(s) = \bar{h}(\bar{W}^{t,\phi}, X^{t,\phi}, Z^{t,\phi}) + \int_s^T \bar{F}(r, X^{t,\phi}(r), Z^{t,\phi}(r), X_r^{t,\phi}) dr \\ \quad - \int_s^T Z^{t,\phi}(r) dW(r), \quad s \in [t, T], \\ X^{t,\phi}(s) = X^{s,\phi}(s), \quad \pi^{t,\phi}(s) = 0, \quad s \in [0, t). \end{array} \right. \quad (52)$$

Using Theorem 7 we see that, under suitable assumptions on the coefficients, there exists a unique solution $(X^{t,\phi}, \pi^{t,\phi})_{(t,\phi) \in [0,T] \times \mathbb{A}}$ of the above system.

From the results of the previous sections, in particular from theorem 14, we have the following representation for the solution (X, Z) of the backward component in system (52). In particular we have that, for every $(t, \phi) \in [0, T] \times \mathbb{A}$,

$$X^{t,\phi}(s) = u(s, \bar{W}^{t,\phi}), \quad \text{for all } s \in [t, T],$$

where $u(t, \phi) = X^{t,\phi}(t)$ is a viscosity solution of the following path-dependent PDE:

$$\left\{ \begin{array}{l} -\partial_t u(t, \phi) - \frac{1}{2} \partial_{xx}^2 u(t, \phi) - \bar{F}(t, u(t, \phi), \partial_x u(t, \phi), (u(\cdot, \phi))_t) = 0, \quad \phi \in \mathbb{A}, \quad t \in [0, T), \\ u(T, \phi) = \bar{h}(\phi, (u(\cdot, \phi))), \quad \phi \in \mathbb{A}. \end{array} \right.$$

An example of this type has been developed first in [7] and then treated by many authors, see, e.g. [20], where they considered a case where the drift and the diffusion component of

the price equation depend both on the wealth process X and the underlying process S . This would lead to a fully coupled forward–backward system which does not fit in our setting. On the other hand in our model we have assumed that the drift μ and the interest rate r may be influenced also by past values of the wealth process X , whereas in cited papers no delay in the backward component is assumed.

5.2 Risk measures via g-expectations

A key problem in financial mathematics is the risk management of an investment. Such a problem has been widely studied in finance since the introductory paper [2] where the notion of *risk measure* has been first introduced. Since then, several empirical studies have been done concerning the key task of risk-management, showing in particular that the best way to quantify the risk of a given financial position should be a *dynamic risk measure*, rather than a classic static one. Starting from this fact the notion of *g-expectation* has been first introduced in [32], as a fundamental mathematical tool if one is to deal with *dynamic risk measure*, we refer also to [34, 38] for a comprehensive and exhaustive introduction to *dynamic risk measures*.

The main purpose of a risk measure is to quantify in a single number the riskness of a given financial position. The next one is the mathematical formulation of the notion of *risk measure*, see, e.g. [11, Definition 13.1.1].

Definition 15 A family $(\rho_t)_{t \in [0, T]}$ of mapping $\rho_t : L^2(\Omega, \mathcal{F}_T) \rightarrow L^2(\Omega, \mathcal{F}_t)$, such that $\rho_T(\xi) = -\xi$ is called *dynamic risk measure*.

From a practical point of view, denoting by ξ the terminal value of a given financial position, $\rho_t(\xi)$ quantifies the risk the investor takes in the position ξ at terminal time T . Clearly, in order to have a concrete financial use, a risk measure has to satisfies a set of properties, usually referred to as *axioms of risk measures*, we refer to [11] to a complete list of the aforementioned axioms.

From a mathematical point of view, it has been shown that BSDE's, and the related forward–backward system, are a perfect tool to tackle the problem of risk management. In particular, one possible way to define a *dynamic risk measure*, is to specify the generator g of the driving BSDE, from here the name *g-expectation*, where the generator g determines the properties of the *dynamic risk measure*. A direct approach to *g-expectation* is therefore to introduce a BSDE of the form

$$Y(t) = \xi + \int_t^T g(s, Y(s), Z(s)) ds - \int_t^T Z(s) dW(s), \quad (53)$$

where the generator g is called the generator of a *g-expectation*. In this sense we have that

$$\rho_t(\xi) = \mathbb{E}^g[\xi | \mathcal{F}_t] = Y(t),$$

where we have used the subscript g to emphasize the role played by the generator g . Heuristically speaking we have the relation

$$\mathbb{E}^g[dY(t) | \mathcal{F}_t] = -g(s, Y(s), Z(s)), \quad 0 \leq t \leq T,$$

so that intuitively the coefficient g reflects the agent's belief on the expected change of risk.

Once we have chosen a risk measure g , such that it is financially reasonable, we then solve the BSDE (53) endowed with a suitable final condition which represents the investor's wealth at terminal time T , we refer to [3, 38] for a detailed introduction to the usage of BSDE as *dynamic risk measures*.

In literature it has always been considered a generator g that depends only on the present value at time t of the risk measure $Y(t)$ and its variability $Z(t)$, but, as pointed out in [10], if we want to model an investor preferences we cannot leave aside the memory effect, that is it is reasonable to assume that an investor makes his choices based also on what happened on the past. In [10] the author proposed to consider a g -expectation which incorporates a disappointment effect through a BSDE of moving average. In fact in the just mentioned work the author suggests that when dealing investor's preferences, to consider Markovian systems is restrictive since it is natural that an investor takes into account the past history of a given investment when he is to make some choice. We refer to [10] for a short but exhaustive review of different economical studies of how memory effect cannot be neglected when dealing with an investor's choice. Regarding the case considered in [10], we make the assumption that the investor has a *short memory*, that is, in making his choices he just consider what has happened in the recent past.

This leads to consider a g -expectation of the form $g(s, y, z) = \beta \bar{y}$, where \bar{y} is the time-average in a sufficiently small time interval and $\beta \in \mathbb{R}$ a given financial parameter, that is we will be dealing with a BSDE with delayed generator of the form

$$Y(t) = \xi + \frac{\beta}{\delta} \int_t^T \int_{-\delta}^0 Y(s+r) dr ds - \int_t^T Z(s) dW(s), \quad (54)$$

with ξ the terminal payoff of the investment to be better introduced in a while and δ small enough such that equation (12) holds.

Let us now assume that the financial market is composed by one risky asset S and one riskless bond B . The generalization to d risky assets can be easily derived from the present case. We assume, in a complete generality, that both the bond and the asset may exhibit delay. For the case of delay in the forward component a more established theory exists, with existence and uniqueness as well as regularity results, see, e.g. [1, 21, 22].

We consider in what follows the delayed market model introduced in [5]. In what concerns the stock price we assume that S evolves according to the following stochastic delay differential equation:

$$\begin{cases} \frac{dS(t)}{S(t)} = \mu(t, S) dt + \sigma(t, S) dW(t), \\ S_0 = s_0 \in \mathbb{R}, \end{cases} \quad (55)$$

where $\mu, \sigma : [0, T] \times \mathbb{A} \rightarrow \mathbb{R}$ are some given functions, where the notation is introduced in Section 1.

Let us assume that μ and σ satisfy assumptions of type (A_1) – (A_2) , so that there exists a unique solution of equation (55) satisfying estimates (10) (this is a consequence of Theorem 4).

Also we assume the investor subscribe a claim with terminal payoff $h : \mathbb{A} \rightarrow \mathbb{R}$ so that

the BSDE (54) becomes

$$Y(t) = h(S_T) + \frac{\beta}{\delta} \int_t^T \int_{-\delta}^0 Y(s+r) dr ds - \int_t^T Z(s) dW(s), \quad (56)$$

where we assume h to satisfy assumption $(A_7) - (iv)$, also let us stress that the generator $g : L^2([-\delta, 0]; \mathbb{R}) \rightarrow \mathbb{R}$ defined above satisfies assumptions $(A_6), (A_7)$, see, e.g. Remark 10. We are naturally led to consider the following forward-backward system with delay

$$\begin{cases} S^{t,\phi}(s) = \phi(t) + \int_t^s S^{t,\phi}(r) \mu(t, S^{t,\phi}) dr + \int_t^s S^{t,\phi}(r) \sigma(t, S^{t,\phi}) dW(r), & s \in [t, T], \\ S^{t,\phi}(s) = \phi(s), & s \in [0, t], \\ Y(s) = h(S^{t,\phi}) + \frac{\beta}{\delta} \int_s^T \int_{-\delta}^0 Y^{t,\phi}(r+\theta) d\theta dr - \int_s^T Z^{t,\phi}(r) dW(r), \\ Y^{t,\phi}(s) = Y^{s,\phi}(s), & s \in [0, t], \end{cases} \quad (57)$$

and by theorems 4–7 we have that the forward-backward system (57) admits a unique solution.

Let us also stress that the great majority of possible claim that can be considered in finance satisfies above assumptions on the terminal payoff h , also we allow the option to possibly be *path-dependent*, that is its terminal value at time T depends explicitly on past values assumed by the asset S .

From the results of the previous sections, we thus have the following characterization for the FBSDE (57). In particular we obtain that theorem 14 holds, so that, for every $(t, \phi) \in [0, T] \times \mathbb{A}$,

$$Y^{t,\phi}(s) = u(s, S^{t,\phi}), \quad \text{for all } s \in [t, T],$$

where $u(t, \phi) = Y^{t,\phi}(t)$ is a viscosity solution of the following path-dependent PDE:

$$\begin{cases} \partial_t u(t, \phi) + \frac{1}{2} \sigma^2(t, \phi) \partial_{xx}^2 u(t, \phi) + \phi \mu(t, \phi) \partial_x u(t, \phi) \\ \quad = \frac{\beta}{\delta} \int_{-\delta}^0 u(t+r, \phi) dr, & \phi \in \mathbb{A}, t \in [0, T], \\ u(T, \phi) = h(\phi), & \phi \in \mathbb{A}. \end{cases}$$

Acknowledgement

The fourth named author gratefully acknowledges the support of the 2010 PRIN project: "Equazioni di evoluzione stocastiche con controllo e rumore al bordo".

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